Math 20100
Calculus I

Lesson 05
The Squeeze Theorem

Dr. A. Marchese, The City College of New York

Bookmarks have been added to this video at the following times:

The Squeeze Theorem:

If $f(x) \leq g(x) \leq h(x)$ for all x near a, except possibly at a, and if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$,

then $\lim_{x \to a} g(x) = L$.

Ex. To find $\lim_{x \to 0} |x| \sin^2\left(\frac{1}{x}\right)$,

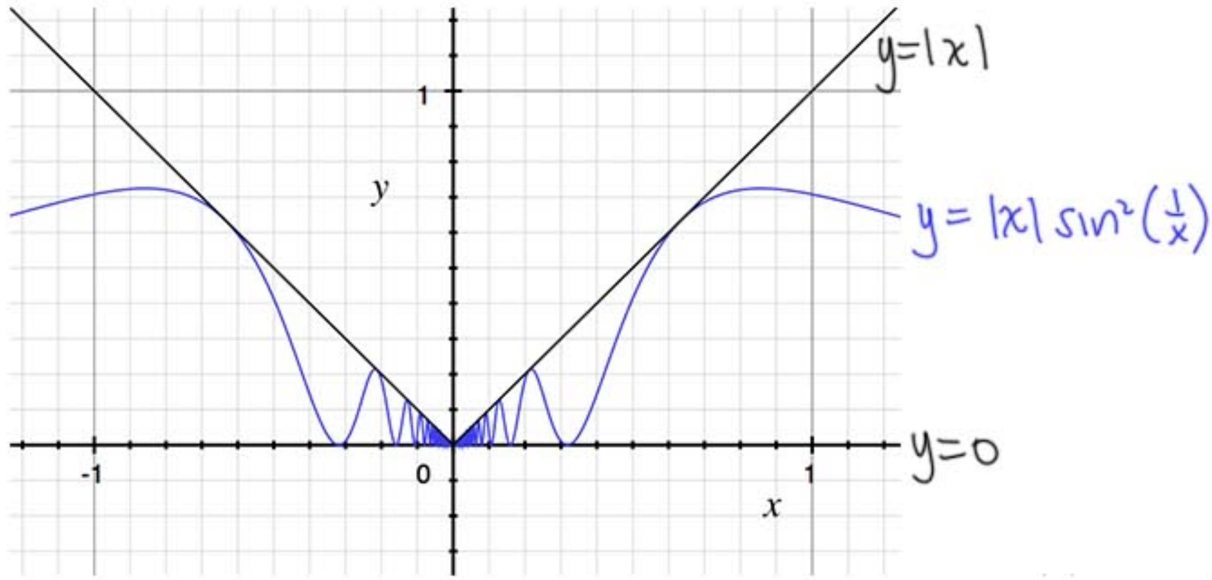
notice $\quad 0 \leq \sin^2\left(\frac{1}{x}\right) \leq 1 \quad$ for all $x \neq 0$

since $|x| > 0$ for all $x \neq 0$, the inequality is preserved by multiplication:

$$0 \leq |x| \sin^2\left(\tfrac{1}{x}\right) \leq |x|$$

$\lim_{x \to 0} 0 = 0 \quad$ and $\quad \lim_{x \to 0} |x| = 0$

$\therefore$ by the Squeeze Theorem, $\lim_{x \to 0} |x| \sin^2\left(\frac{1}{x}\right) = 0$.



Now let's revisit $\lim_{x \to 0} \frac{\sin x}{x}$.

In lesson 3, we found this limit numerically:

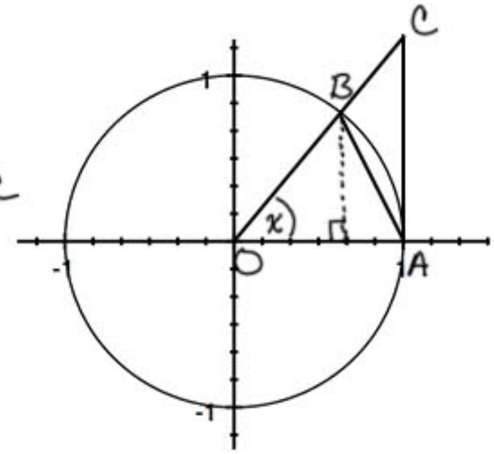
	$x \to 0^-$				$x \to 0^+$		
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
sinx/x	0.9983342	0.9999833	0.9999998		0.9999998	0.9999833	0.9983342
	$y \to 1$				$y \to 1$		

Now, we can prove this algebraically using the Squeeze Theorem.

We are going to show that $\cos x \leq \frac{\sin x}{x} \leq 1$ for x near 0 then since $\lim\limits_{x\to 0} \cos x = \cos(0) = 1$ and $\lim\limits_{x\to 0} 1 = 1$,

we'll have $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

Consider angle x with $0 < x < \frac{\pi}{2}$



area $\triangle AOB \le$ area sector $AOB \le$ area $\triangle AOC$

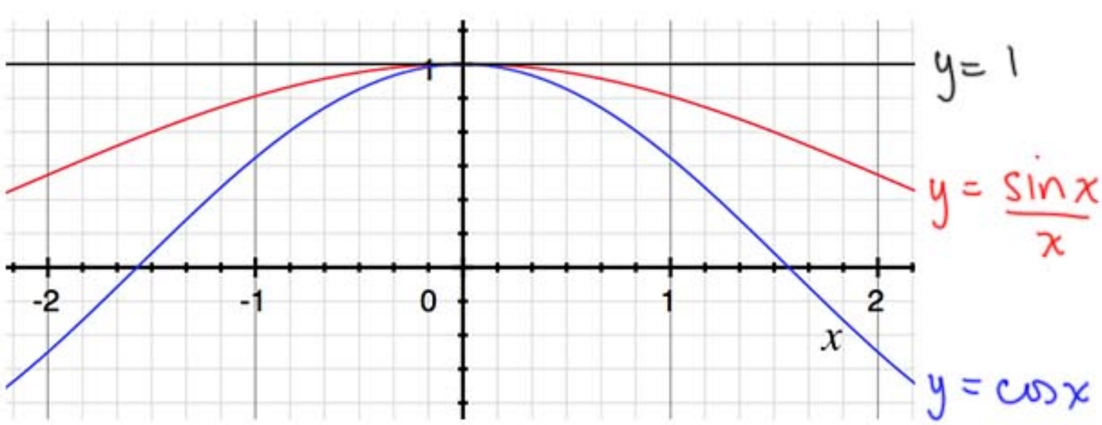
$\frac{1}{2}(1)(\sin x) \le \frac{x}{2}(1)^2 \le \frac{1}{2}(1)\tan x$

(area of sector of $\measuredangle\theta$, radius r is $\frac{\theta}{2}r^2$)

multiply by $\frac{2}{\sin x}$ ($\sin x > 0$ for $0 < x < \frac{\pi}{2}$)

$$1 \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

taking reciprocals, $\cos x \le \dfrac{\sin x}{x} \le 1$.



$$\therefore \lim_{x \to 0} \frac{\sin x}{x} = 1$$

Why is this better than the numerical method?

By taking limits numerically, we are approximating.

Using algebraic methods, we are computing exactly.

Note: $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1 \Rightarrow \lim\limits_{kx \to 0} \frac{\sin(kx)}{kx} = 1$

and $\lim\limits_{x \to 0} \frac{x}{\sin x} = 1 \Rightarrow \lim\limits_{kx \to 0} \frac{kx}{\sin(kx)} = 1$.

Ex. $$\lim_{x \to 0} \frac{\sin 4x}{x} = \lim_{x \to 0} \frac{4 \sin 4x}{4x}$$

$$= 4 \lim_{\underline{x} \to 0} \frac{\sin 4x}{4x} = 4 \lim_{\underline{4x} \to 0} \frac{\sin 4x}{4x} = 4(1) = 4.$$

Ex. $\displaystyle \lim_{t\to 0} \frac{\sin 2t}{\sin 6t} = \lim_{t\to 0} \frac{\sin 2t}{1} \cdot \frac{1}{\sin 6t}$

$$= \lim_{t\to 0} \frac{2}{6} \frac{\sin 2t}{2t} \frac{6t}{\sin 6t}$$

$$= \frac{2}{6} \lim_{t\to 0} \frac{\sin 2t}{2t} \cdot \frac{6t}{\sin 6t}$$

$$= \frac{2}{6} \left(\lim_{t\to 0} \frac{\sin 2t}{2t}\right)\left(\lim_{t\to 0} \frac{6t}{\sin 6t}\right)$$

$$= \frac{2}{6} \left(\lim_{2t\to 0} \frac{\sin 2t}{2t}\right)\left(\lim_{6t\to 0} \frac{6t}{\sin 6t}\right) = \frac{2}{6}(1)(1) = \frac{2}{6}.$$

Ex. $\displaystyle \lim_{\theta\to 0} \frac{\theta + \tan\theta}{\sin\theta} = \lim_{\theta\to 0} \frac{\theta}{\sin\theta} + \lim_{\theta\to 0} \frac{\tan\theta}{\sin\theta}$

* if they both exist

$$= \lim_{\theta\to 0} \frac{\theta}{\sin\theta} + \lim_{\theta\to 0} \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\sin\theta}$$

$$= 1 + 1 = 2.$$