Math 20100
Calculus I

Lesson 4
Calculating Limits

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Calculating Limits

left limit = right limit, real number.

Limit Laws: if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist,

1) $\lim_{x \to a} \left(f(x) + g(x)\right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

the limit of the sum is the sum of the limits.

2) $\lim_{x \to a} \left(f(x) - g(x)\right) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

the limit of the difference is the difference of the limits.

3) $\lim_{x \to a} \left(c\, f(x)\right) = c \left(\lim_{x \to a} f(x)\right)$

the limit of a constant times a function is equal to the constant times the limit of the function.

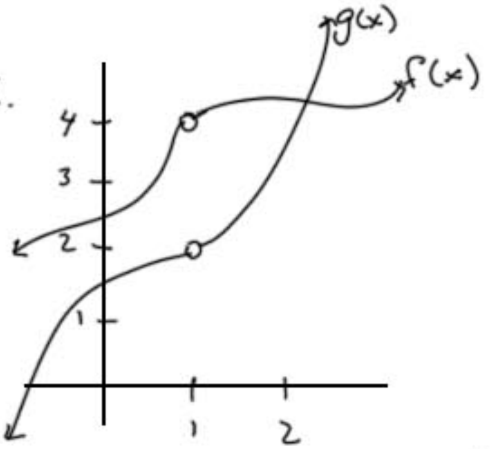
4) $\lim_{x \to a} \left(f(x)\, g(x)\right) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right)$

the limit of a product is the product of the limits.

5) $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ provided $\lim_{x \to a} g(x) \neq 0$

the limit of the quotient is the quotient of the limits.

Ex.



here we have

$$\lim_{x\to 1} f(x) = 4 \quad \text{and} \quad \lim_{x\to 1} g(x) = 2$$

we don't have graph of $y = f(x)+g(x)$

1. $\lim_{x\to 1}\left(f(x)+g(x)\right) = \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x) = 4+2 = 6$

2. $\lim_{x\to 1}\left(f(x)-g(x)\right) = \lim_{x\to 1} f(x) - \lim_{x\to 1} g(x) = 4-2 = 2.$

3. $\lim_{x\to 1} 5f(x) = 5\lim_{x\to 1} f(x) = 5\cdot 4 = 20.$

4. $\lim_{x\to 1} f(x)g(x) = \left(\lim_{x\to 1} f(x)\right)\left(\lim_{x\to 1} g(x)\right) = 4\cdot 2 = 8$

5. $\lim_{x\to 1} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to 1} f(x)}{\lim_{x\to 1} g(x)} = \dfrac{4}{2} = 2.$

More Limit Laws: given $\lim_{x\to a} f(x)$ exists

6) $\boxed{\lim_{x\to a} \left(f(x)\right)^n = \left(\lim_{x\to a} f(x)\right)^n}$

Ex. above

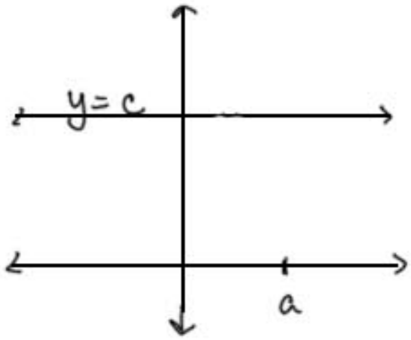
$\lim_{x\to 1} \left(f(x)\right)^3 =$

$\left(\lim_{x\to 1} f(x)\right)^3 = 4^3 = 64.$

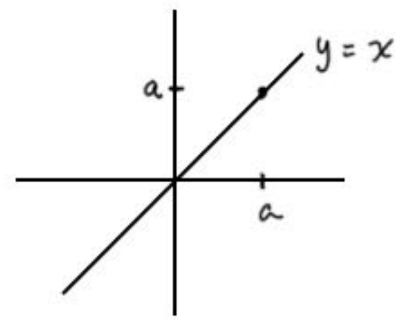
7) $\boxed{\lim_{x \to a} c = c}$

8) $\boxed{\lim_{x \to a} x = a}$

think of line $y = c$:



line $y = x$:



9. $\boxed{\lim_{x \to a} x^n = a^n}$ (from laws 8 + 6)

n is a positive integer

10. $\boxed{\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}}$

11. $\boxed{\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}}$

Notice: two ideas with laws 6–11

1) we can pull/push the limit through some operations.

$$\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n, \quad \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$

We see many more operations (functions) for which this is true in Lesson 5.

2) We can just plug in $x = a$:

$$\lim_{x\to a} x = a \quad , \quad \lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$$

This is called "direct substitution" to find the limit. We see more of this later in this lesson, and more in Lesson 5.

Ex. Given $\lim_{x\to 2} f(x) = 1 \quad \lim_{x\to 2} g(x) = 0 \quad \lim_{x\to 2} h(x) = 5$

1. $$\lim_{x\to 2}\left(3f(x) + h(x)\right) = \lim_{x\to 2}\left(3f(x)\right) + \lim_{x\to 2} h(x)$$

$$= 3\lim_{x\to 2} f(x) + \lim_{x\to 2} h(x) = 3(1) + 5 = 8.$$

2. $$\lim_{x\to 2} 2\left(g(x) + h(x)\right)^2 = 2\lim_{x\to 2}\left(g(x) + h(x)\right)^2$$

$$= 2\left[\lim_{x\to 2}\left(g(x) + h(x)\right)\right]^2 = 2\left(\lim_{x\to 2} g(x) + \lim_{x\to 2} h(x)\right)^2$$

$$= 2(0+5)^2 = 2(25) = 50.$$

The Direct Substitution Property:

If $f(x)$ is a polynomial function,

a rational function,

or a trig function

with $x = a$ in its domain, then

$$\boxed{\lim_{x \to a} f(x) = f(a)}.$$

Ex. $\displaystyle \lim_{x \to \frac{\pi}{2}} \sin x = \sin\left(\tfrac{\pi}{2}\right) = 1$

Ex. $\displaystyle \lim_{x \to 3} \left(2x^2 - x + 1\right) = 2(3)^2 - 3 + 1 = 18 - 3 + 1 = 16.$

Ex. $\displaystyle \lim_{x \to 1} \frac{3x-1}{x^2+1} = \frac{3(1)-1}{1^2+1} = \frac{2}{2} = 1.$

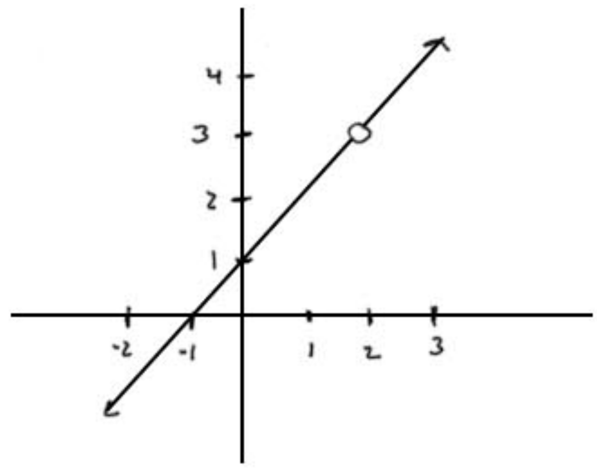
Note That The Direct Substitution Property requires $x=a$ to be in The domain of the function.

What about something like:

$$f(x) = \frac{x^2 - x - 2}{x-2}$$

$$= \frac{(x-2)(x+1)}{x-2}$$

$$= x+1 \quad \text{for } x \neq 2$$



looking at $\lim\limits_{x \to 2} \frac{x^2-x-2}{x-2}$ we can't plug in $x=2$.

but we see from The graph $\lim\limits_{x \to 2} \frac{x^2-x-2}{x-2}$ exists, $= 3$.

We can simplify, Then take the limit:

$$\lim_{x \to 2} \frac{x^2-x-2}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+1)}{\cancel{x-2}} = \lim_{x \to 2} x+1 = 2+1 = 3$$

the limits are equal even Though The functions

direct substitution

are not (They have different domains)

Note: for $x = 2$, $\dfrac{(x-2)(x+1)}{x-2}$ does not exist.

but $\dfrac{(x-2)(x+1)}{x-2} = x+1$ for all $x \neq 2$.

and when we take $\lim_{x \to a} f(x)$, we never care about what happens <u>at</u> $x=a$, we only care about what happens as $\underbrace{x \text{ approaches } a}_{x \neq a}$.

So for limits, $\lim_{x \to 2} \dfrac{(x-2)(x+1)}{x-2} = \lim_{x \to 2} x+1$

because $x \neq 2$ in either case.

$\therefore$ If $f(x) = g(x)$ for $x \neq a$, then

$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ provided the limit exists.

Ex. $\lim_{x \to -1} \dfrac{2x^2+3x+1}{x^2-2x-3}$ try direct substitution (plug in $x=-1$) get $\frac{0}{0}$

so we factor

$$\lim_{x \to -1} \frac{2x^2+3x+1}{x^2-2x-3} = \lim_{x \to -1} \frac{(2x+1)\cancel{(x+1)}}{\cancel{(x+1)}(x-3)} = \lim_{x \to -1} \frac{2x+1}{x-3}$$

$$= \frac{2(-1)+1}{-1-3} = \frac{-1}{-4} = \frac{1}{4}.$$

Ex. $\lim_{h \to 0} \dfrac{(4+h)^2-16}{h}$ try plugging in $h=0$, get $\frac{0}{0}$

so we expand and simplify

$$= \lim_{h \to 0} \frac{\cancel{16}+8h+h^2-\cancel{16}}{h} = \lim_{h \to 0} \frac{8h+h^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(8+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} 8+h = 8+0 = 8.$$

Ex. $\lim_{u \to 2} \dfrac{\sqrt{4u+1}-3}{u-2}$ if the limit in the denominator were not zero, we could plug in by laws 5, 2, and 11. so we rationalize the numerator and simplify, then take the limit

$$\lim_{u\to 2} \frac{\sqrt{4u+1}-3}{u-2} = \lim_{u\to 2} \left(\frac{\sqrt{4u+1}-3}{u-2}\right)\left(\frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}\right)$$

$$= \lim_{u\to 2} \frac{\left(\sqrt{4u+1}\right)^2 - \cancel{3\sqrt{4u+1}} + \cancel{3\sqrt{4u+1}} - 9}{(u-2)\left(\sqrt{4u+1}+3\right)}$$

$$= \lim_{u\to 2} \frac{4u+1-9}{(u-2)\left(\sqrt{4u+1}+3\right)} = \lim_{u\to 2} \frac{4u-8}{(u-2)\left(\sqrt{4u+1}+3\right)}$$

$$= \lim_{u\to 2} \frac{4\cancel{(u-2)}}{\cancel{(u-2)}\left(\sqrt{4u+1}+3\right)} = \lim_{u\to 2} \frac{4}{\sqrt{4u+1}+3} \quad \text{plug in } u=2$$

$$= \frac{4}{\sqrt{4(2)+1}+3} = \frac{4}{3+3} = \frac{4}{6} = \frac{2}{3}.$$

Ex. $\displaystyle \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$ again, can't just plug in so we combine & simplify first

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2}\cdot\frac{x^2}{x^2} - \frac{1}{x^2}\cdot\frac{(x+h)^2}{(x+h)^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}$$

$$= \lim_{h\to 0} \frac{x^2-(x+h)^2}{(x+h)^2x^2h} = \lim_{h\to 0} \frac{x^2-(x^2+2xh+h^2)}{(x+h)^2x^2h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{(x+h)^2 x^2 h} = \lim_{h\to 0} \frac{-2xh - h^2}{(x+h)^2 x^2 h}$$

plug in $h=0$

$$= \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{(x+h)^2 x^2 \cancel{h}} = \lim_{h\to 0} \frac{-2x-h}{(x+h)^2 x^2} = \frac{-2-0}{(x+0)^2 x^2} = \frac{-2}{x^4}.$$

Using Left and Right Limits

For limits involving piecewise functions, we often need to check left and right limits separately.

Ex. $f(x) = \begin{cases} 2-x^2 & x \le -1 \\ x+4 & -1 < x < 2 \\ x^2+2 & x \ge 2 \end{cases}$

find $\lim_{x\to -1} f(x)$ and $\lim_{x\to 2} f(x)$, if They exist.

For $\lim_{x\to -1} f(x)$, we consider $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} 2-x^2 = 2-(-1)^2 = 1$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x+4 = -1+4 = 3$$

Since $\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$, $\lim_{x \to -1} f(x)$ DNE.

For $\lim_{x \to 2} f(x)$, consider $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$

Work on this problem
on your own

$$f(x) = \begin{cases} 2-x^2 & x \leq -1 \\ x+4 & -1 < x < 2 \\ x^2+2 & x \geq 2 \end{cases}$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x+4 = 2+4 = 6$$

same

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2+2 = 2^2+2 = 6$$

$\therefore \lim_{x \to 2} f(x)$ exists and $= 6$.

For limit expressions involving absolute value, we need to express the absolute value in its piecewise notation, and check left and right limits separately.

Ex. $\lim\limits_{x \to 5} \dfrac{|x-5|}{3x-15}$

we know

$$|x-5| = \begin{cases} x-5 & \text{for } x-5 \geq 0,\ x \geq 5 \\ -(x-5) & \text{for } x-5 < 0,\ x < 5 \end{cases}$$

So $$\lim_{x \to 5^-} \frac{|x-5|}{3x-15} = \lim_{x \to 5^-} \frac{-(x-5)}{3x-15} = \lim_{x \to 5^-} \frac{-\cancel{(x-5)}}{3\cancel{(x-5)}} =$$

$$= \lim_{x \to 5^-} -\frac{1}{3} = -\frac{1}{3}.$$

And $$\lim_{x \to 5^+} \frac{|x-5|}{3x-15} = \lim_{x \to 5^+} \frac{x-5}{3x-15} = \lim_{x \to 5^+} \frac{\cancel{x-5}}{3\cancel{(x-5)}} = \lim_{x \to 5^+} \frac{1}{3} = \frac{1}{3}.$$

$$\therefore \lim_{x \to 5^-} \frac{|x-5|}{3x-15} \neq \lim_{x \to 5^+} \frac{|x-5|}{3x-15} \text{ and } \lim_{x \to 5} \frac{|x-5|}{3x-15} \text{ DNE.}$$

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