Math 20100
Calculus I

Lesson 3
An Introduction to Limits

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An Introduction to Limits

$$\lim_{x \to a^-} f(x) = L$$

"the limit of $f(x)$ as x approaches a from the left is equal to L"

says as x approaches a from values that are less than a, $\underbrace{f(x)}_{y}$ approaches L.

$$\lim_{x \to a^+} f(x) = L$$

"the limit of $f(x)$ as x approaches a from the right is equal to L"

says as x approaches a from values that are greater than a, $\underbrace{f(x)}_{y}$ approaches L.

$$\lim_{x \to a} f(x) = L$$

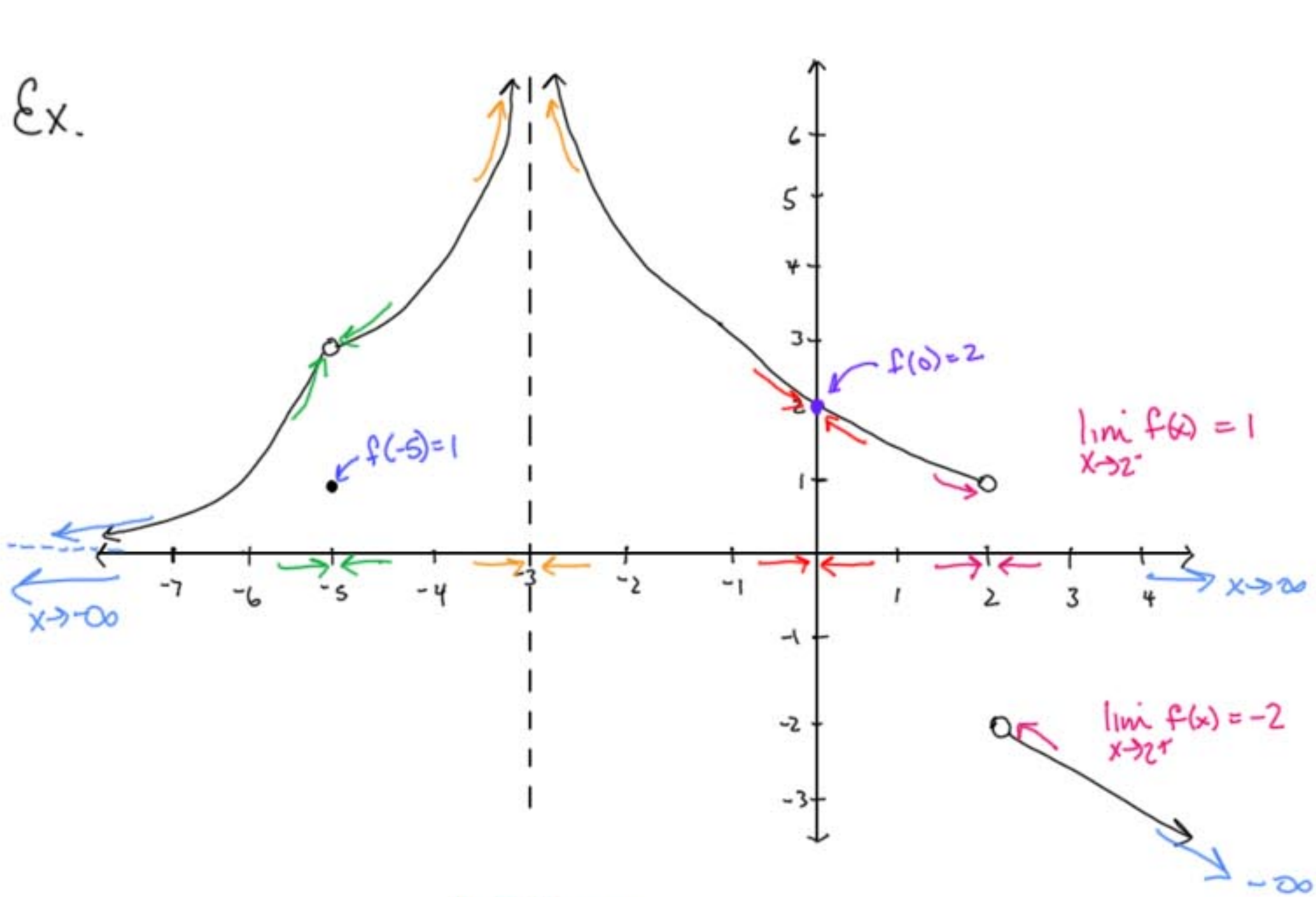
"the limit of $f(x)$ as x approaches a is equal to L"

means

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

as x approaches a (from both sides), $\underbrace{f(x)}_{y}$ approaches L.

Ex.



$$\lim_{x\to 0} f(x) = 2 \qquad \lim_{x\to 0^-} f(x) = 2 \qquad \lim_{x\to 0^+} f(x) = 2 \qquad f(0) = 2.$$

no + or −, means two sided limit

$$\lim_{x\to -5} f(x) = 3 \qquad \lim_{x\to -5^-} f(x) = 3 \qquad \lim_{x\to -5^+} f(x) = 3 \qquad f(-5) = 1$$

two-sided limit

$$\lim_{x\to -3} f(x) = \infty$$

two-sided limit

* Note this limit does not exist as a real number. But instead of writing DNE, writing ∞ gives more information

$\lim_{x \to 2} f(x)$ DNE does not exist because $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to \infty} f(x) = -\infty$ again, This limit DNE as a real number, but writing $-\infty$ gives more information than writing DNE

Note: Although these limits do not exist as real numbers,

we say $\lim_{x \to a} f(x) = \infty$ when the values of $f(x)$ increase without bound as x approaches a. (can also be a one-sided limit).

And $\lim_{x \to a} f(x) = -\infty$ when the values of $f(x)$ decrease without bound as x approaches a. (can also be a one-sided limit).

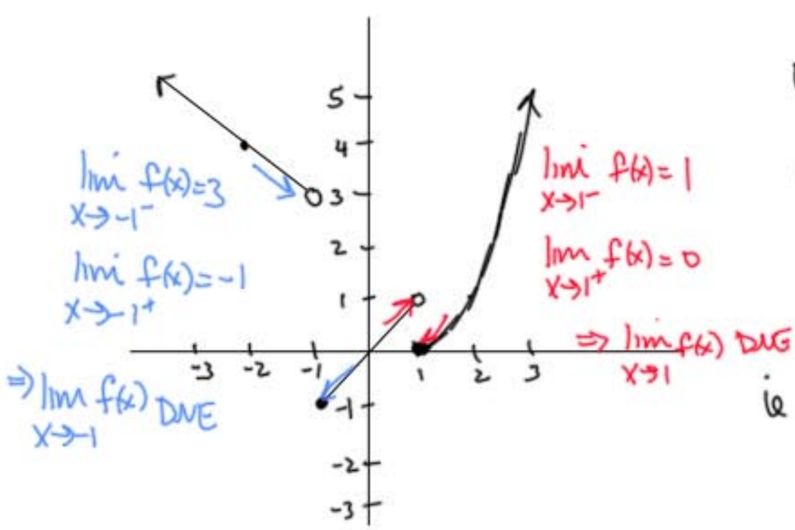
Ex. Sketch the graph of $f(x)$, and use it to determine the values for a for which

$\lim_{x \to a} f(x)$ exists.

limit is the same from both sides, and is a real number (not infinite)

$$f(x) = \begin{cases} 2-x & x < -1 \\ x & -1 \le x < 1 \\ (x-1)^2 & x \geq 1 \end{cases}$$

STOP Work on this problem on your own



We see that the limit does not exist (DNE) for: $a = -1$, $a = 1$

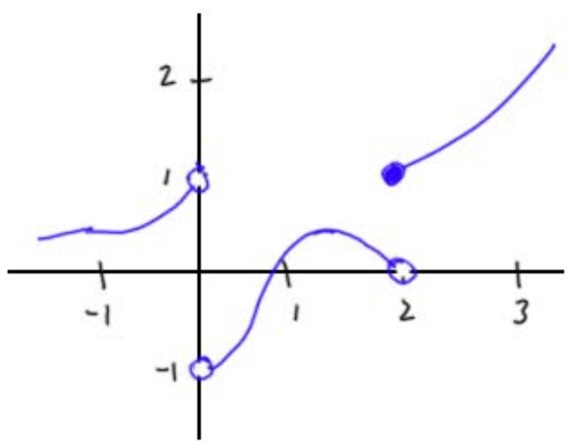
ie $\lim_{x \to -1} f(x)$ DNE and $\lim_{x \to 1} f(x)$ DNE.

$\therefore$ The limit exists for $a \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Ex. Sketch a function $f(x)$ satisfying:

$\lim_{x\to 0^-} f(x) = 1$ $\quad \lim_{x\to 0^+} f(x) = -1$ $\quad f(0)$ undefined

$\lim_{x\to 2^-} f(x) = 0$ $\quad \lim_{x\to 2^+} f(x) = 1$ $\quad f(2) = 1$



Finding limits numerically:

$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$

to find $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$x\to 0^-$ $\quad\quad x \to 0^+$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
sinx/x	0.9983342	0.9999833	0.9999998		0.9999998	0.9999833	0.9983342

$y \to 1$ $\quad\quad y \to 1$

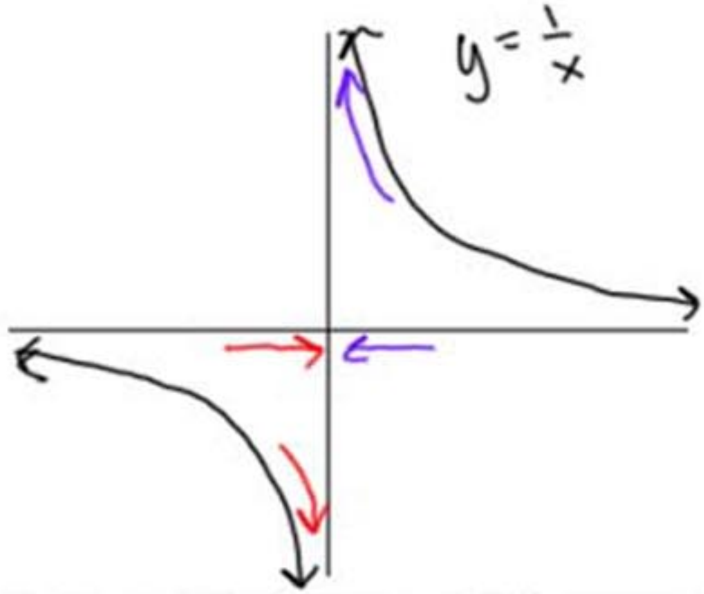
Ex. $\lim_{x \to 2} \frac{x^2-2x}{x^2-4} = \frac{1}{2}$

	$x \to 2^-$				$x \to 2^+$		
x	1.9	1.99	1.999	2	2.001	2.01	2.1
(x^2-2x)/(x^2-4)	0.4871795	0.4987469	0.499875		0.500125	0.5012469	0.5121951
	$y \to \frac{1}{2}$				$y \to \frac{1}{2}$		

Ex. $\lim_{x \to 0} \frac{1}{x}$ DNE

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$ $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

	$x \to 0^-$				$x \to 0^+$		
x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
1/x	-10	-100	-1000		1000	100	10
	$y \to -\infty$				$y \to +\infty$		



* dividing by zero

When you see $\frac{\text{nonzero}}{\text{zero}}$, Think $\pm\infty$

infinite behavior

(* but $\frac{\text{zero}}{\text{zero}}$ an indeterminate form
means need more info)

Here we're introduced limits graphically and numerically. In lesson 4 we learn to take limits algebraically, as well.