Math 20100
Calculus I

Lesson 1
Functions

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Functions

When one quantity depends on another, we can represent this relationship as a <u>function</u>.

Ex. If temperature depends on time

↑ dependent variable (depends on time)

↑ independent variable (does not depend on any other quantity)

Let T = temperature

t = time

we can write $T = f(t)$

ie, Temperature is a function of time.

In general, x = independent variable

y = dependent variable

$y = f(x)$ y is a function of x.

Note that the defining characteristic of a

Def: <u>function</u> is that for each value of the independent variable, there is one and only one corresponding value for the dependent variable

Ex. For each time t above, there is one and only one temperature T.

In general, for each x-value, there is one and only one corresponding y-value.

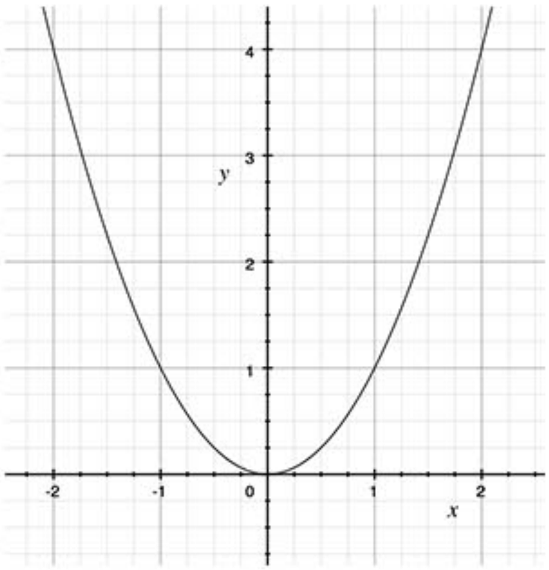
Representations of Functions:

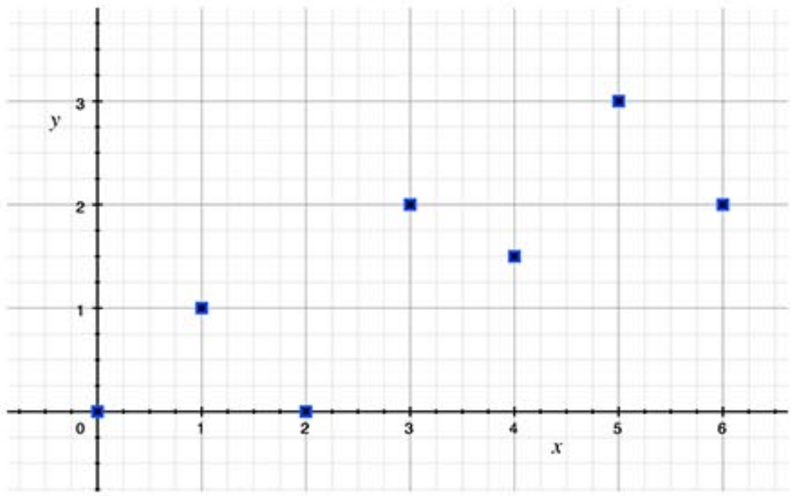
We typically see functions represented in the following three ways:

1. algebraically   Ex. $y = 2x^2 + 5x - 3$

   or $f(x) = 2x^2 + 5x - 3$

2. graphically





3. numerically (table of data)

x	$f(x)$
0	0
1	1
2	0
3	2
4	1.5
5	3
6	2

With the algebraic representation, it is important to understand proper function evaluations:

Ex. $f(x) = 2x^2 - 3x + 6$

find $f(0) = 2(0)^2 - 3(0) + 6 = 6$

$f(2) = 2\underbrace{(2)^2}_{} - 3(2) + 6$

$2 \cdot 4$

$8 - 6 + 6 = 8$

$f(a) = 2a^2 - 3a + 6$

$f(x+h) = 2(x+h)^2 - 3(x+h) + 6$

$= 2(x^2 + 2xh + h^2) - 3x - 3h + 6$

$= 2x^2 + 4xh + 2h^2 - 3x - 3h + 6$

$$\frac{f(x+h) - f(x)}{h} = \frac{(2x^2 + 4xh + 2h^2 - 3x - 3h + 6) - (2x^2 - 3x + 6)}{h}$$

$$= \frac{\cancel{2x^2}+4xh+2h^2-\cancel{3x}-3h\cancel{+6}-\cancel{2x^2}+\cancel{3x}-\cancel{6}}{h}$$

$$= \frac{4xh+2h^2-3h}{h} = \frac{\cancel{h}(4x+2h-3)}{\cancel{h}}$$

$$= 4x+2h-3.$$

$$f(x) = 2x^2-3x+6$$

$$\frac{f(x)-f(a)}{x-a} = \frac{(2x^2-3x+6)-(2a^2-3a+6)}{x-a}$$

$$= \frac{2x^2-3x\cancel{+6}-2a^2+3a-\cancel{6}}{x-a}$$

$$= \frac{2x^2-3x-2a^2+3a}{x-a}$$

$$= \frac{2(x^2-a^2)-3(x-a)}{x-a}$$

$$= \frac{2(x+a)(x-a)-3(x-a)}{x-a}$$

$$= \frac{\cancel{(x-a)}\left[2(x+a)-3\right]}{\cancel{x-a}}$$

$$= 2(x+a)-3.$$

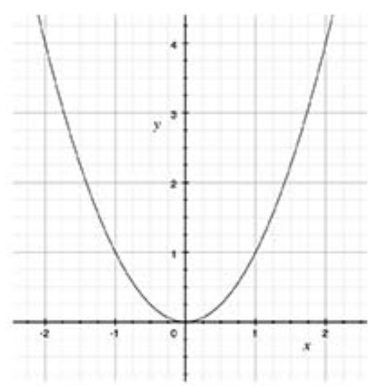
The <u>domain</u> of $y=f(x)$ is the set of x-values we can plug into $f(x)$.

The <u>range</u> of $y=f(x)$ is the set of y-values that come from $f(x)$.

Ex. $f(x)=x^2$ Domain: $\mathbb{R}$ (all real numbers)

$y=x^2$ $(-\infty,\infty)$

Range: $[0,\infty)$



More formal notation:

Domain: $\{x\in\mathbb{R}\}$

Range: $\{y\in\mathbb{R} \mid y\geq 0\}$

Tips for finding the domain (when given the algebraic form):

1) look out for fractions, because the denominator $\neq 0$.

2) look out for even roots, because what is under the even root must not be negative.

(3) in calc II, look out for logs.)

Ex. $f(x) = \dfrac{1}{x+2}$ we know
$$\begin{aligned} x+2 &\neq 0 \\ -2 \quad & \quad -2 \\ x &\neq -2. \end{aligned}$$

domain: $\{x \in \mathbb{R} \mid x \neq -2\}$ or $x \neq -2$.

Ex. $g(x) = \sqrt{5-x}$ we know
$$\begin{aligned} 5-x &\geq 0 \\ +x \quad & \quad +x \\ 5 \geq x &\Rightarrow x \leq 5 \end{aligned}$$

domain: $\{x \in \mathbb{R} \mid x \leq 5\}$ or $x \leq 5$

or $(-\infty, 5]$

Ex. $h(x) = \dfrac{\sqrt[4]{x+1}}{x-1}$

we need x such that $x+1 \geq 0$ AND $x-1 \neq 0$

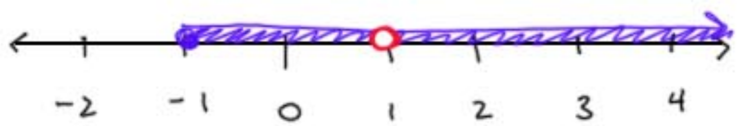
$\qquad\qquad\qquad\qquad -1 \quad -1 \qquad\qquad\qquad +1 \quad +1$

$\underbrace{x \geq -1}$ AND $\underbrace{x \neq 1}$

↑

intersection of sets

helps to use number line notation:



domain: $[-1, 1) \cup (1, \infty)$.

Ex. $f(x) = \sqrt{x^2 + x}$ we need $x^2 + x \geq 0$

factor $x(x+1) \geq 0$

either:

$x \geq 0$ AND $x+1 \geq 0$

$\qquad\qquad\qquad -1 \quad -1$

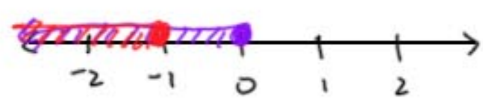
$\underbrace{x \geq 0}$ AND $\underbrace{x \geq -1}$

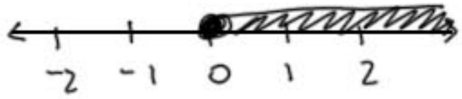
OR

$x \leq 0$ AND $x+1 \leq 0$

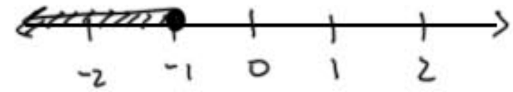
$\qquad\qquad\qquad -1 \quad -1$

$\underbrace{x \leq 0}$ AND $x \leq -1$

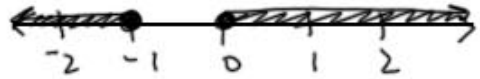




OR



$x \geq 0$ $\qquad\qquad$ $x \leq -1$



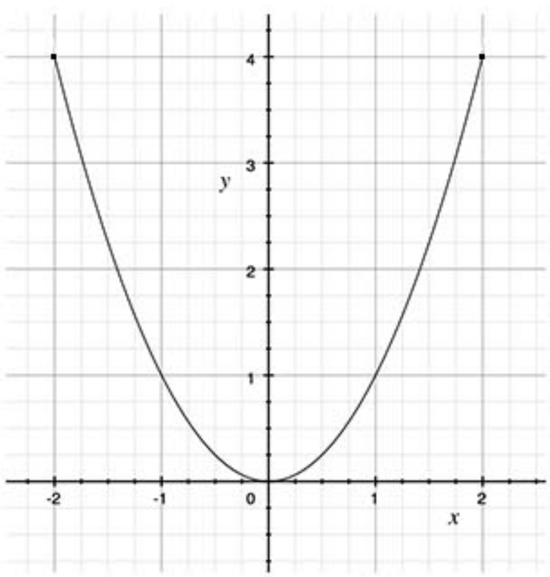
$(-\infty, -1] \cup [0, \infty)$

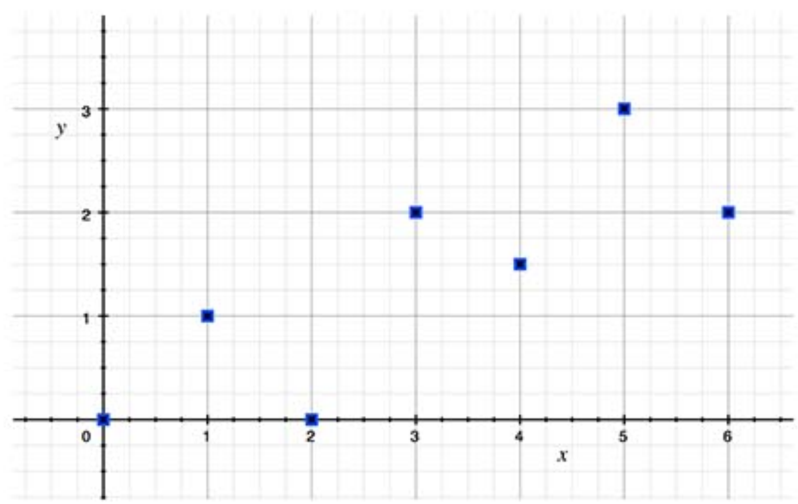
domain: $\{x \in \mathbb{R} \mid x \geq 0 \text{ or } x \leq -1\}$

For functions given graphically or numerically:

for the domain, find the set of x-values shown.

for the range, find the set of y-values shown.





Domain: $\{x \in \mathbb{R} \mid -2 \leq x \leq 2\}$

$[-2, 2]$

Range: $\{y \in \mathbb{R} \mid 0 \leq y \leq 4\}$

$[0, 4]$

Domain: $\{0, 1, 2, 3, 4, 5, 6\}$

Range: $\{0, 1, 1.5, 2, 3\}$.

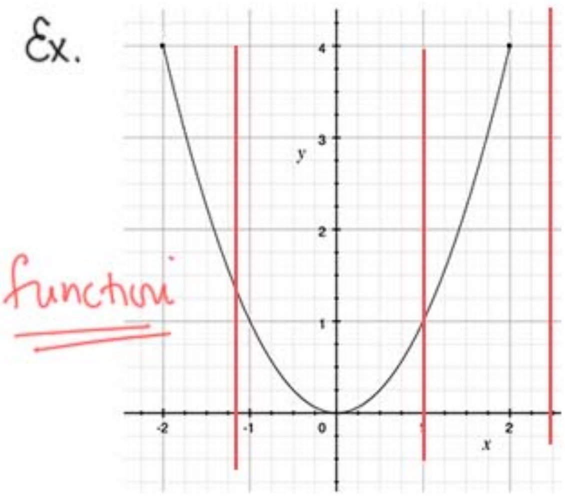
x	$f(x)$
0	0
1	1
2	0
3	2
4	1.5
5	3
6	2

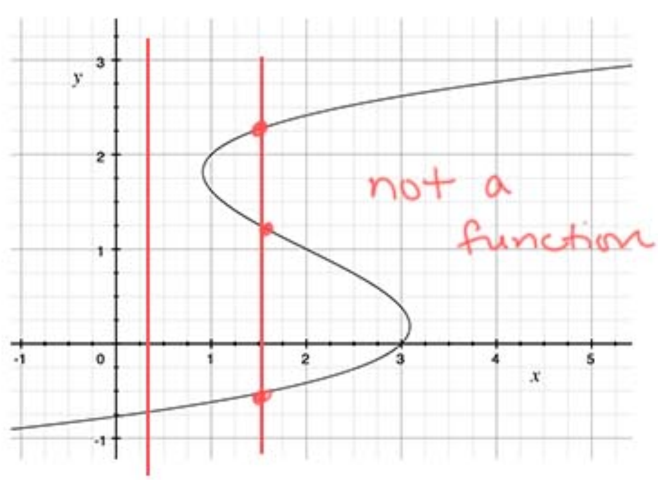
Domain: $\{0,1,2,3,4,5,6\}$

Range: $\{0,1,1.5,2,3\}$.

Note that when we are given a graph, we can check to see if it represents a function $y=f(x)$ by using <u>the vertical line test</u>: if the graph represents a function $y=f(x)$, any vertical line will intersect the graph at most once.

Ex.





Piecewise Functions:

Ex. $f(x) = \begin{cases} 3 - \frac{1}{2}x & x \le 2 \\ 2x - 5 & x > 2 \end{cases}$

to evaluate a particular x-value, need to determine which part of the function to use.

$f(0) = 3 - \frac{1}{2}(0) = 3$

$\uparrow$ $0 \le 2$ use top function $3 - \frac{1}{2}x$

$f(2) = 3 - \frac{1}{2}(2) = 3 - 1 = 2$

$\uparrow$ $2 \le 2$ use top function $3 - \frac{1}{2}x$

$f(4) = 2(4) - 5 = 8 - 5 = 3$

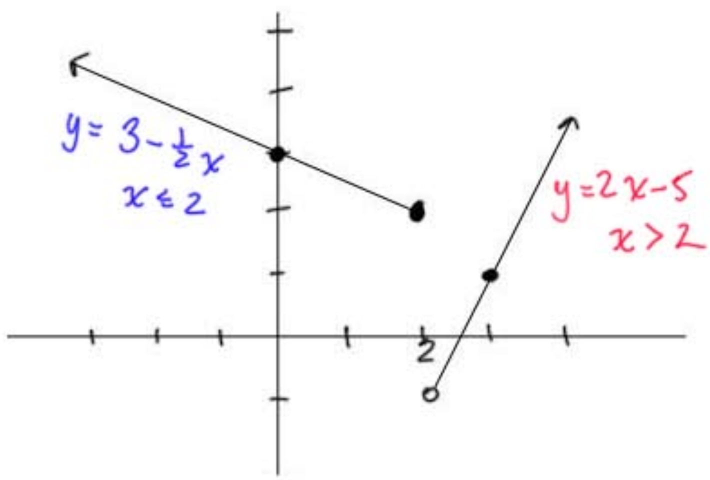
$\uparrow$ $4 > 2$ use bottom function $2x - 5$

To find the domain of a piecewise function, take the union of the individual domains

Ex. $\left. \begin{array}{l} x \le 2 \\ x > 2 \end{array} \right\}$ together these are all reals.

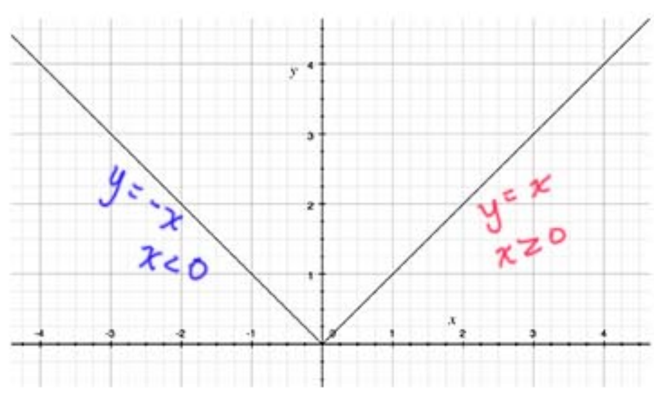
To graph a piecewise function, graph each part for its own domain (on same axes)

Ex. $f(x) = \begin{cases} 3 - \frac{1}{2}x & x \le 2 \\ 2x - 5 & x > 2 \end{cases}$



Note: The absolute value function can be expressed as a piecewise function:

$f(x) = |x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$



Ex. $|2| = 2 \quad 2 \ge 0$

$|-5| = -(-5) = 5 \quad -5 < 0$

Symmetry:

<u>Def.</u> $f(x)$ is <u>even</u> if $f(-x) = f(x) \quad \forall x$ in domain

$f(x)$ is <u>odd</u> if $f(-x) = -f(x) \quad \forall x$

"for all" math notation

Ex. $f(x) = x^2$ is $f(x)$ even, odd, neither?

plug in $-x$ and simplify:

$$f(-x) = (-x)^2 = (-x)(-x) = x^2 = f(x)$$

$\therefore \; f(-x) = f(x)$ and $f(x)$ is <u>even</u>.

Ex. $f(x) = x^3$ is $f(x)$ even, odd, neither?

$$f(-x) = (-x)^3 = (-x)\overbrace{(-x)(-x)}^{+} = -x^3 = -f(x)$$

$\therefore f(-x) = -f(x)$ and $f(x)$ is <u>odd</u>.

Ex. $f(x) = x|x|$ even, odd, or neither?

$$f(-x) = -x|-x| = -x|x| = -f(x)$$

$\therefore$ $f(x)$ is <u>odd</u>.

Ex. $f(x) = 1 + 3x^3 - x^5$ even, odd, or neither?

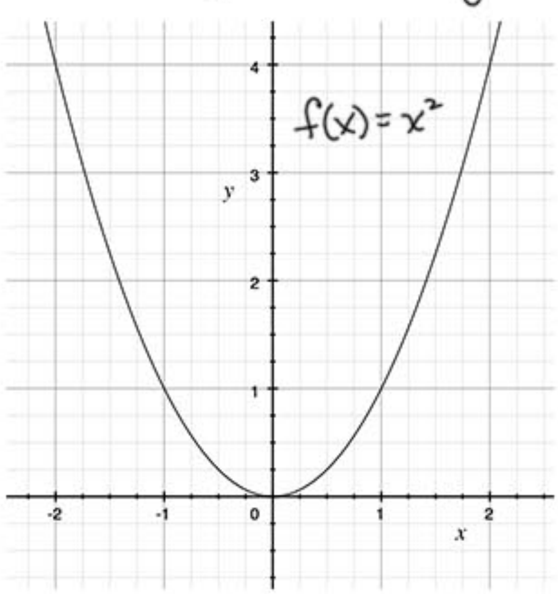
$$f(-x) = 1 + 3(-x)^3 - (-x)^5$$

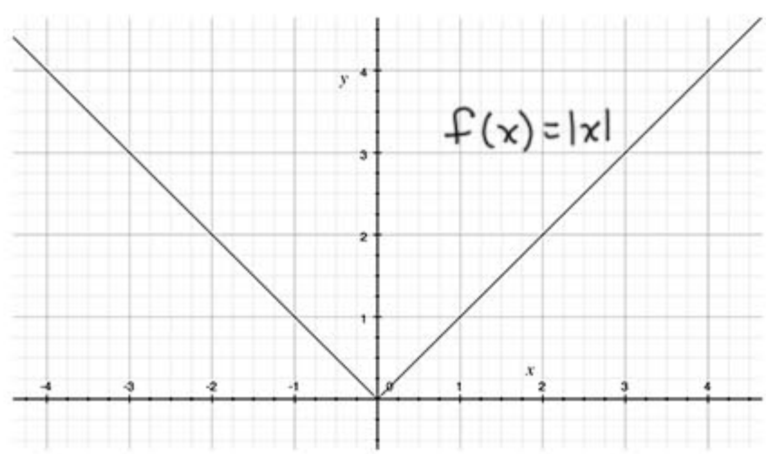
$$= 1 - 3x^3 + x^5 \neq f(x)$$

$$\neq -f(x)$$

$f(x)$ is neither.

Graphs of even functions have symmetry through the y-axis:





Graphs of odd functions have symmetry through the origin: