Introduction to **Simplicial Dynamical Systems**

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Let $f : X \to X$ be a continuous map with X a compact metric space. We define a dynamical system by iterating such a function on X. A computational analysis of the system is an attempt to describe its behavior by using a finite set of data points with the individual points themselves subject to round off error.

This is a well studied problem, see especially Hsu (1987). The most direct approach is to choose a finite subset F of X, regarding it as a finite approximation of the space. For each point $z \in F$ choose $g(z)$, a point of F close to $f(z)$, so that the map g on F is regarded as an approximation of f on X. In effect, we apply f to each point of F and then round off to obtain values in F. Alternatively, we can choose a finite cover F of X by small, nonempty subsets of X, "cells" or "pixels". For each set $z \in F$ the image $f(z)$ meets one or more elements of F. This defines a relation on the finite set F which we regard as the approximation to f, see Hsu (1987), Akin (1993) Chapter 5 and Osipenko (to appear). In this monograph we introduce a third mode of approximating the function f using finite data.

Consider the simpler problem of sketching the graph of a real-valued function defined on a bounded interval. We partition the domain by small subintervals using a finite, increasing sequence of points. At each such point we evaluate the function and so compute, up to roundoff, the corresponding sequence of points on the graph. If we connect successive points by a line segment then we get the graph of a piecewise linear function which approximates the original. This function obtained by linear interpolation really carries no more information than did the finite sequence of points from which it was determined. However, it has the advantage that it is the same sort of object as that which we intended to study. It is a real-valued function on the original domain, an especially simple one because it is piecewise linear.

In our original problem we will restrict to the case where X is a compact polyhedron and we will approximate the function f on X by special simplicial maps.

We will use as references: Rourke and Sanderson (1972), Hudson (1969) and Stallings (1968). These describe a polyhedron as a special compact subset of a Euclidean space. Our results easily extend to objects in the PL category, that is, compact metric spaces equipped with a PL structure, a class of p.l. compatible homeomorphisms to polyhedra. From there they apply to smooth manifolds.

The simplices of a simplicial complex K triangulating a polyhedron X

are subsets of X. Furthermore, each has a linear structure as a closed convex subset of the ambient Euclidean space.

Now suppose that a triangulation K of X consists of a small simplices. That is, the mesh of K is small, where the mesh is the maximum among the diameters of the simplices z of K. Let K' be the barycentric subdivision of K. Each simplex z of K has a regular neighborhood $N(z, K')$ the union of the simplices of K' which intersect z. Think of the z's or the $N(z, K')$'s as the cells of our second approximation method.

The image set $f(z)$ may stretch and bend in X, usually encountering many simplices of K. However, if K^* is a fine enough subdivision of K then for each $z^* \in K^*$ $f(z^*)$ is contained in the interior of $N(z, K')$ for some $z \in K$. Furthermore, it is possible to estimate how small the mesh of K^* has to be so that this condition holds.

If the image of each z^* is contained in the interior of some $N(z, K')$ then we denote the smallest such z by $s_f(z^*)$. It is easy to check that if z_1^* is a face of z^* in K^* then $s_f(z_1^*)$ is a face of $s_f(z^*)$ in K. By mapping the barycenter of z^* to the barycenter of $s_f(z^*)$ we define a simplicial map $g : K^{*\prime} \to K'$. If K^* was in fact a subdivision of K' then we have obtained from f a simplicial map $g : L^* \to L$ where $L^* = K^{*\prime}$ is a subdivision of $L = K'$. The vertices of L^* correspond to the finite set of points in our first approximation method. A simplicial map is completely determined by linearity once the vertex values are known. This is the analogy with our piecewise linear curve sketch.

A subdivision K^* of a simplicial complex K is called a *proper* subdivision if no simplex of K^* meets disjoint simplices of K. This is a very mild condition. If K^* is any subdivision of the barycentric subdivision of K then it is a proper subdivision of K. By a *simplicial dynamical system* on a polyhedron X we mean a triangulation K of X, a proper subdivision K^* of K and a simplicial map g from K^* to K. We use the same symbol for both the mapping of infinite sets of simplices, $g : K^* \to K$, and for the underlying piecewise linear mapping g on X. We described above a way of approximating the dynamical system f by a simplicial dynamical system. The approximating map g on X will be called a *p.l. roundoff map* for f.

This monograph analyzes the behavior of simplicial dynamical system. Because $K^* \neq K$ we cannot directly iterate the simplicial map $g : K^* \to K$. If $z^* \in K^*$ then $z = g(z^*) \in K$ and so usually contains many simplices of K^*. Thus, we can define a multiple valued function, or relation, G^* on the finite set K^*: z_1^* is G^* related to z^* if $z_1^* \subset g(z^*)$. There is a similar relation

G on K: z_1 is G related to z if $z_1 \subset g(z)$. The recurrence properties of the p.l. map g on X can be computed by using the finite relations G^* and G.

As an approximation procedure p.l. roundoff, like linearization, consists of an easy part and a hard part. The easy part is setting up the approximation and analyzing the approximating object. After all if this part were not tractable the approximation would be pointless. This easy part is what the rest of this little book does in detail. Before summarizing the individual chapters, we will say a few words about the hard part: getting back to the original system from the approximate one.

The p.l. roundoff map g approximates f in the C^0 sense. To be precise, given $\epsilon > 0$ there exist $\delta, \delta^* > 0$ so that if $\mathrm{mesh}(K) < \delta$ and the $\mathrm{mesh}(K^*) < \delta^*$ where K^* is a subdivision of K', then the simplex association $s_f(z^*) \in K$ is defined for all $z^* \in K^*$ and the p.l. roundoff map g on X satisfies $d(f(x), g(x)) \leq \epsilon$ for all $x \in X$. Thus, in general, these methods detect properties which are robust enough to be preserved by C^0 approximation, see Akin (1993) Chapter 7. For this reason we focus our attention below on chain recurrence and basic sets. By imposing special conditions on f one can go farther. If g is ϵ close to f then a g orbit is an ϵ chain for f, also called an ϵ pseudo-orbit for f. By imposing hyperbolicity assumptions on f and using shadowing theorems one can sometimes show that the g orbits approximate true f orbits.

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Chapter 1, Chain Recurrence and Basic Sets: We review from Akin (1993) some of the notation and results for dynamics of a closed relation F on a compact metric space X. Of greatest importance is the *chain recurrent set* $|\mathcal{C}F|$ and the *basic sets* contained therein, i.e. the $\mathcal{C}F \cap (\mathcal{C}F)^{-1}$ equivalence classes, also called the chain components. In particular, we describe from Miller and Akin (1996) results comparing the recurrence properties of F with those of the two-sided shift homeomorphism s_F and the one-sided shift map s_F^+ on the associated sample path spaces $X_F \subset X^{\mathbf{Z}}$ and $X_F^+ \subset X^{\mathbf{Z}_+}$, respectfully. These have two different applications.

If F is itself a continuous map on X then s_F^+ is conjugate to F itself and s_F is the shift homeomorphism on the inverse limit of the system $\dots X \xrightarrow{F} X \xrightarrow{F} X$. On the other hand, if X is a finite set, thought of as an alphabet, then F is the set of admissible dipthongs and s_F and s_F^+ are the associated subshifts of finite type.

Of special interest is the *two alphabet case.* Given two finite sets A^* and A suppose that $g : A^* \to A$ is a map and $J : A^* \to A$ is a relation. The relations $G^* = J^{-1} \circ g$ on A^* and $G = g \circ J^{-1}$ on A have closely related dynamics. Thus, when A^* is a much larger set than A we can use G to study G^*.

When f is a homeomorphism on X a *pre-decomposition* $\tilde{\mathcal{F}}$ for f is a finite collection of closed, nonempty, invariant subsets of X such that the positive and negative limits sets of any orbit sequence for f are each contained in elements of $\tilde{\mathcal{F}}$. By concatenating we obtain the associated decomposition which is a pre-decomposition by pairwise disjoint sets. We review from Akin (1993) how the basic sets are obtained from decompositions (Proposition 2.5).

Chapter 2, Simplicial Maps and Their Local Inverses: We review the definitions and notations for simplicial complexes (always assumed finite) and the compact polyhedra they triangulate. When a complex K triangulates a polyhedron X, we write $X = |K|$ and define on X a metric d_K by using the l' to compare barycentric coordinates.

Between complexes K^* and K, a simplicial map $g : K^* \to K$ is a map of finite sets with special properties associated with the complex structure. e.g. g preserves the incidence. There is an associated piecewise linear map, denoted $g : |K^*| \to |K|$, which is linear on each simplex of K^*.

If $g : K^* \to K$ is a simplicial map and z^* is a simplex of K^* with $z = g(z^*)$ in K then the dimensions satisfy dim $Z \leq$ dim z^*. We call z^* *degenerate* if the inequality is strict and so z^* is *nondegenerate* when dim $z =$ dim z^*.

If z^* is nondegenerate then the restriction $g : z^* \to z$ is a linear isomorphism and so admits an inverse map $\overline{g}_{z^*} : z \to z^*$. When z^* is degenerate we can define a family of one-sided inverse maps. For each vertex v of z choose a point of $z^* \cap g^{-1}(v)$. The set of such choices is a convex cell we denote T^{z^*}. So z^* is nondegenerate when T^{z^*} is a single point. For each $t \in T^{z^*}$ we define a map $\overline{g}_{z^*,t} : z \to z^*$ by extending linearly from the vertex choices.

Suppose that K^* is a subdivision of K. We call K^* a *proper subdivision* of K if no simplex of K^* meets disjoint simplices of K. For example, if K' is the barycentric subdivision of K then K' and any further subdivision $K*$ of K' are proper for K.

A *simplicial dynamical system* is a simplicial map $g : K^* \to K$ where K^* is a proper subdivision of K. The main result of this chapter is:

Theorem: *Let $g : K^* \to K$ be a simplicial dynamical system. For $z^* \in K^*$ and $t \in T^{z^*}$ the maps $\overline{g}_{z^*,t} : g(z^*) \to z^*$ contract the metric d_K uniformly. That is, the supremum of the Lipschitz constants is less than 1.*

Chapter 3, The Shift Factor Maps for a Simplicial Dynamical System: We define the maps which are used to study a simplicial dynamical system $g : K^* \to K$. Since K and its proper subdivision K^* triangulate a common polyhedron X, the associated p.l. map g defines a topological dynamical system in X.

First, let K^* be the union of the pairs $\{z^*\} \times T^{z*}$ as z^* varies over K^* with $\hat{p} : \hat{K} \to K^*$ the projection defined by $(z^*, t) \mapsto z^*$. K^* and K are finite sets, but $\hat{K}$ is a union of a finite collection of pairwise disjoint, compact, convex sets.

For $x \in X$ there are defined *carriers* $q^*(x), q(x)$ and $\hat{q}(x)$ in K^*, K and $\hat{K}$. The simplex of K^* whose simplex interior contains x is $q^*(x)$. Similarly, for $q(x)$. $\hat{q}(x)$ is the unique pair $(z^*, t) \in \hat{K}$ such that $z^* = q^*(x)$ and x is in the image denoted $\langle z^*, t\rangle$ the local inverse map $\overline{g}_{z^*,t}$.

We use the *inclusion relation* $J = \{(z^*, z) \in K^* \times K : z^* \subset z\}$ to regard $g : K^* \to K$ and $J : K^* \to K$ as an example of the two alphabet described in Chapter 1 with relations $G^* = J^{-1} \circ g$ on K^* and $G = g \circ J^{-1}$ on K. We define $\hat{G}$ on $\hat{K}$ to be $(\hat{p}\times\hat{p})^{-1}(G^*)$ so that $(z_1^*, t_1) \in \hat{(G)}(z_0^*, t_0)$ iff $z_1^* \in G^*(z_0^*)$.

The shifts $s^+_{G^*}$ on $K^{*+}_{G^*} \subset K^{\mathbf{Z}_+}$ and $s^+_{\hat{G}}$ on $\hat{K}^+_{\hat{G}} \subset \hat{K}^{\mathbf{Z}_+}$ are related to the p.l. map g on X by the functions $q^+ : X \to K^{*+}_{G^*}$ and $\hat{q}^+ : X \to \hat{K}^+_{\hat{G}}$ defined by $q^+(x)_i = q^*(g^i(x))$ and $\hat{q}^+(x)_i = \hat{q}(g^i(x))$ for all $i \in \mathbf{Z}_+$. However, q^+ and $\hat{q}^+$ are not continuous. The central object of our study is a map going the other way.

$$\hat{h}^+ = \{((z^*, t), x) \in \hat{K}^+_{\hat{G}} \times X : g^i(x) \in \langle z^*, t\rangle \text{ for all } i \in \mathbf{Z}_+\}$$

defines a continuous function from $\hat{K}^+_{\hat{G}}$ to X mapping the shift $s^+_{\hat{G}}$ to g. The closed relation $h^+ \circ (\hat{p})^{-1} \subset K^{*+}_{G^*} \times X$ restricts to a function on various important invariant subsets of $K^{*+}_{G^*}$. The proof that $\hat{h}^+$ is a function uses the contraction results of the previous chapter. They also yield a Partial Shadowing Lemma: If $\epsilon > 0$, $(z^*, t) \in \hat{K}^+_{\hat{G}}$ and $\chi \in X^{\mathbf{Z}_+}$ are such that $\chi_i \in \langle z^*_i, t_i \rangle$ and $d_K(g(\chi_i), \chi_{i+1}) \leq \epsilon$ for all $i \in \mathbf{Z}_+$ then with $x = \hat{h}^+(z^*, t)$ $d_K(\chi_i, g^i(x)) \geq C\epsilon$ for all $i \in \mathbf{Z}_+$ where the constant C depends only on the simplicial map g.

Because we can explicitly describe when two elements of $\hat{K}^+_{\hat{G}}$ are mapped to the same point of X by $\hat{h}^+$, we obtain various semiconjugacy results. For example, suppose that K^*_1 is a subdivision of K *isomorphic* to K^*. That is, there is a simplicial isomorphism $r_1 : K^*_1 \to K^*$ such that $r(z) = z$ for all $z \in K$. Then $g_1 = g \circ r_1 : K^*_1 \to K$ is a simplicial dynamical system. There exists a homomorphism ρ_1 on X such that $\rho_1(z) = z$ for all $z \in K$ and $\rho_1 \circ g_1 = g \circ \rho_1$. That is, the topological dynamical systems associated with the p.l. maps g_1 and g on X are conjugate. However, ρ_1 is *not* the p.l. map associated with r_1. In fact, ρ_1 is usually not p.l. at all.

Chapter 4, Recurrence and Basic Set Images: Let $g : K^* \to K$ be a simplicial dynamical system with $X = |K|$. From the two alphabet results of Chapter 1 we associate to each basic set B^* for the relation $F^* = J^{-1} \circ g$ on K^* a basic set B for $G = g \circ J^{-1}$ on K and vice-versa. The simplices of B^* and of B all have the same dimension. We call B^* and B k *skeleton basic sets* when this common dimension is k.

When B^* is a basic set for G^* then the relation h^+ restricts to a continuous map on $B^{*+}_{G^*} = B^{*\mathbf{Z}_+} \cap K^{*+}_{G^*}$. We call $h^+(B^{*+}_{G^*})$ the associated *basic set image* in X. The restriction of g to each basic set image is a topologically transitive map with dense periodic points. Conversely, if x is a recurrent point for g on X then the sequence of carriers: $q^*(x), /; q^*(g(x))$,... lies entirely in a single basic set for G^* called the *endset* of x and x lies in the corresponding basic set image. Thus, the union of the basic set images for g is the *Birkhoff center* of g, i.e. the closure of the set of recurrent points. From the basic set images one can construct the–usually larger–basic sets for g and the entire chain recurrent set for g. Each basic set image can be constructed using a procedure mimicking the *iterated function system* of Barnsley.

Let B^* be a k skeleton basic set for G^* with B the associated basic set for G. It is always true that $h^+(B^{*+}_{G^*}) \subset |B^*| \subset |B|$. If these inclusions are equalities, and so the basic set image is the k dimensional polyhedron $|B|$, then B^* is called a *polyhedral* basic set. Otherwise both inclusions are strict and $h^+(B^{*+}_{G^*})$ is nowhere dense in $|B|$. In that case B^* is called *tattered*. If $h^+(B^{*+}_{G^*})$ meets the interior of some k simplex of B then B^* is called an *interior* basic set. So polyhedral implies interior but there exist interior basic sets which are tattered. The endsets are all interior and every basic set image is contained in some interior basic set image.

The interior basic sets are of special interest because if B^* is interior then the restriction of h^+ to $B^{*+}_{G^*}$ is an *almost homeomorphism* onto its image. That is, for x in a dense G_δ subset of $h^+(B^{*+}_{G^*})$ the set $(h^+)^{-1}(x) \cap B^{*+}_{G^*}$ consists of a single point, namely $q^+(x)$.

When g is a nondegenerate simplicial map, i.e. every $z^* \in K^*$ is nondegenerate and so $\hat{p} : \hat{K} \to K^*$ is a bijection, then there are special results. Suppose X is *everywhere d dimensional*, i.e. $X = |S^d(K^*)|$ where $S^d(K^*) = \{z^* : \dim z^+ = d\}$. Then the restriction of h^+ to $S^d(K^*)^+_{G^*} = S^d(K^*)^{\mathbf{Z}_+} \cap K^{*+}_{G^*}$ is an almost homeomorphism onto X mapping the subshift to g.

Chapter 5, Invariant Measures: For g the p.l. map on a polyhedron X associated with the simplicial dynamical system $g : K^* \to K$ ($X = |K|$), the invariant measures come from the shift. That is, if μ is an ergodic invariant measure for g there exists an interior basic set B^* for G^* and an ergodic invariant measure ν for the shift on $B^{*+}_{G^*}$ such that $\mu = h^+_* \nu$. In fact, h^+ is an isomorphism between the measurable dynamical systems $(s^+_{G^*}, B^{*+}_{G^*}, \nu)$ and (g, X, μ). Of particular importance are the measures which come from Markov measures on the shift.

The relation $G^*_{B^*}$ obtained by restricting the relation G^* to the basic set B^* has a *characteristic matrix* from which is obtained a Markov chain on B^* and an associated Markov measure, ν^P, the *Parey measure* on $B^{*+}_{G^*}$. Its entropy is $\ln \gamma$ where γ is the dominant eigenvalue of the characteristic matrix for B^*. If B^* is an interior basic set then $\ln \gamma$ is also the topological entropy of g on the basic set image $h^+(B^{*+}_{G^*})$. The topological entropy of g is obtained by letting B^* vary over the interior basic sets and taking the maximum of the $\ln \gamma$'s.

For each simplex z of K with $\dim z = k$ there is a k dimensional Lebesque measure λ_z on z normalized by $\lambda_z(z) = 1$. If B^* is a polyhedral basic set for G^* with associated G basic set B, there is a unique positive vector p such that the $\sum p_z = 1$ and $\sum p_z \lambda_z = \lambda_B$ (summing over $z \in B$) is an invariant measure for g on $h^+(B^{*+}_{G^*}) = |B|$. The invariant Lebesque measure λ_B is the projection via h^+ of a Markov measure on $B^{*+}_{G^*}$ and is erodic. The set of x in X such that $\{g^i(x)\}$ is eventually in some polyhedral basic set image $|B|$ is residual and intersects each $z \in K$ in a subset of λ_z measure 1.

If K^*_1 is a subdivision of K isomorphic to K^* by the simplicial isomorphism $r_1 : K^*_1 \to K$ and $g_1 = g \circ r_1$: $K^*_1 \to K$ is the isomorphic simplicial dynamical system then the relation G on K is the same for g and g_1, and the relations G^*_1 on K^*_1 and G^* on K^* are related by r_1. If B^* is a polyhedral basic set for G^* then $B^*_1 = r_1^{-1}(B^*)$ is a polyhedral basic set for G^*_1 with the same associated G basic set B. The Chapter 3 conjugacy homeomorphism ρ_1 on X mapping g_1 to g restricts to a homeomorphism of $|B|$. However, ρ_1 does not relate the Lebesque measures λ_1 for g_1 and λ for g on $|B|$. In fact, the measurable systems $(g, |B|, \lambda_1)$ and $(g, |B|, \lambda)$ often have different entropy. As they are different measures ergodic for g λ and $\rho_1 * \lambda_1$ are, in that case, mutually singular. There exists cases where for no isomorph of g does Lebesque measure yield the topological entropy.

Chapter 6, Generalized Simplicial Dynamical Systems: Suppose $g : K^* \to K$ is a simplicial map with K^* a subdivision of K but not a proper subdivision. Although g is not a simplicial dynamical system it can nonetheless happen that $\hat{h}^+ : \hat{K}_{\hat{G}} \to X$ is a function and that all of the earlier results are true for these *generalized simplicial dynamical systems*. In this chapter we illustrate how such systems arise and we characterize them.

Chapter 7, Examples: After the higher dimensional Tent Map (Example 6.1), the examples are all on $L =$ a single 2 simplex together with its faces. We exhibit examples with fractal basic set images and with nonwandering points which are not contained in any basic set image. Also there is an example with entropy zero despite having a basic set shift (but not an interior basic set shift) with positive entropy.

Chapter 8, PL Roundoffs of a Continuous Map: After the details needed for the construction of the p.l. roundoff maps as described above, we consider certain special cases satisfying a general position condition. In these cases the roundoff maps have certain additional structure, a filtration similar to one discussed in Akin, Hurley and Kennedy (1996) Proposition 6.3.

Chapter 9, Nondegenerate Maps on Manifolds: If $g : K^* \to K$ is a simplicial dynamical system and the polyhedron $X = |K|$ is a manifold then there exists a subdivision K_1^* of K^* and a nondegenerate simplicial map $g_1 : K_1^* \to K$ such that the uniform distance between the p.l. maps g and g on X is bounded by 4 times the mesh of K. From this one can prove that for a compact, connected PL manifold X, the set of chain transitive maps on X is a nonempty uniformly closed subset of the space of continuous maps on X and that it contains the set of weak mixing maps as a residual subset.

Appendix: Chapter 10, Stellar and Lunar Subdivisions: We review the simplicial constructions associated with *stellar subdivisions* of a complex and introduce the related idea of a *lunar subdivision.*

Appendix: Chapter 11, Hyperbolicity for Relations: We review the concepts of shadowing and expansivity for homeomorphisms and extend them to closed relations. In particular, we prove that each relation concept is equivalent to the corresponding homeomorphism concept applied to the sample path shift homeomorphism.

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Since the magic in all these results is not very subtle, I will violate the magicians' code and reveal the trick. A simplicial dynamical system $g : K^* \to K$ has built in hyperbolicity. The little simplices of K^* are stretched to fit on simplicies of K. That is the expanding part. The (super-) contracting part consists of the linear singularities of g on the degenerate simplicies. In particular, the nondegenerate cases behave somewhat like expanding maps.

What about more general p.l. maps? Let K be a complex on X and $g : X \to X$ any p.l. map. There exist subdivisions K_1 and K_2 of K such that $g : K_1 \to K_2$ is simplicial. Let K^* be a common subdivision of K_1 and K_2 and for $\alpha = 1, 2$ let $J_\alpha : K^* \to K_\alpha$ be the corresponding inclusion relation. Define the relation G^* on K^* by

$$G^* = J_2^{-1} \circ g \circ J_1.$$

For each point $x \in X$ there exist G^* chains $\mathbf{z}^* \in K^{*\mathbf{Z}_+}$ such that $g^i(x) \in \mathbf{z}_i^*$ for all $i \in \mathbf{Z}_+$. The trouble is that now given $\mathbf{z}^* \in L_{G^*}^{*+}$ such an x need not exist for $\mathbf{z}^*$ and need not be unique when it does exist even in the case when $\mathbf{z}^*$ is periodic.

It might be worth considering such extensions of simplicial dynamical systems because a smooth map on a smooth manifold probably has p.l. approximations which are somewhat better than merely C^0 while the p.l. roundoff maps, degenerate as they are, are inevitably only C^0 close. However, such further considerations are a story for another day.