1. Find the LU factorization of the following matrices A.

   (a) $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -1 \\ -3 & 1 & 1 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix}$.

2. Use the previous LU factorization of the matrices A to solve the systems below. No credit will be given for any other method.

   (a)
$$\begin{aligned} x + 2y - z &= 3 \\ 2x + y - 2z &= 3 \\ -3x + y + z &= -6. \end{aligned}$$

   (b)
$$\begin{aligned} x_1 + x_2 &= 3 \\ 3x_1 - 4x_2 &= 2 \end{aligned}$$

3. Write code to eliminate in the first column of a 100×100 matrix without pivoting and using one loop.

4. Use $PA = LU$ factorization to solve the following systems. No credit will be given for any other method.

   (a)
$$\begin{aligned} 3x + y + 2z &= 0 \\ 6x + 3y + 4z &= 1 \\ 3x + y + 5z &= 3. \end{aligned}$$

   (b)
$$\begin{aligned} 3x_1 + 7x_2 &= 1 \\ 6x_1 + x_2 &= -11 \end{aligned}$$

5. (a) Find the $PA = LU$ factorization of $A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 4 & -4 \\ 1 & 3 & 1 \end{bmatrix}$.

   (b) Solve $Ax = b = \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$ using the above $PA = LU$ factorization.

6. (a) Show that $A = \begin{bmatrix} 1 & 2 \\ 3 & 10 \end{bmatrix}$ is not symmetric.

   (b) Show that $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ is symmetric but is not positive definite.

   (c) Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}$ is symmetric but is not positive definite.

7. Find the Cholesky factorization, $A = GG^T$, of each of the following symmetric, positive definite matrices using the algorithm on p. 116.

   (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$

(b) $A = \begin{bmatrix} 25 & 5 \\ 5 & 26 \end{bmatrix}$

8. Use the previous Cholesky factorization and then forward and back substitution to solve

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 2x_1 + 8x_2 &= 2 \end{aligned}$$

(No credit will be given for any other method.)

9. Find the Cholesky factorization, $A = GG^T$, of the symmetric, positive definite matrices $A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}$ using $G = LD^{\frac{1}{2}}$.

10. Given $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$ and $B = \begin{bmatrix} .0001 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) Roughly describe why matrix A is ill-conditioned and matrix B is well-conditioned.

(b) Use the numpy command *cond* to approximate the norm 2 condition numbers $\kappa_2(A)$ and $\kappa_2(B)$. Does the numpy approximate fit your rough idea above?

11. Find the norms $\|A\|_2 = \lambda_1$ and the condition numbers $\kappa(A) = \frac{\lambda_1}{\lambda_2}$ of the following positive definite matrices (by hand):

(a) $A = \begin{bmatrix} 100 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$.

12. Find the norms $\|A\|_2$ and the condition numbers $\kappa(A)$ of the following positive definite matrices (by hand) from $\sqrt{\lambda_1(A^TA)}$ and $\sqrt{\lambda_n(A^TA)}$:

(a) $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

13. Given the ill-conditioned matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$.

(a) Solve $Ax = b$ for $b = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

(b) Solve $Ax = b$ for $b = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}$.

(c) You should see that the solutions, x, to this ill-conditioned problem are very sensitive to small changes in the input b. There is no robust algorithm.

14. Even well conditioned problems can give errors. Given $B = \begin{bmatrix} .0001 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) Solve $Bx = b$ for $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ using no pivots with three digits of accuracy.

(b) Solve $Bx = b$ for $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ using partial pivoting with three digits of accuracy.

15. The system $\begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3.0001 \end{bmatrix}$ has one solution $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Suppose $\hat{x} = \begin{bmatrix} 3 \\ -0.0001 \end{bmatrix}$ is the approximate solution after running some algorithm.
    (a) Compute the residual $\hat{r} = b - A\hat{x}$, $\|\hat{r}\|_\infty$, and $\|\hat{r}\|_2$.
    (b) Compute the relative error in the residual, $\frac{\|\hat{r}\|_2}{\|b\|_2}$. You may use numpy's LA.norm(r) command.
    (c) Compute the relative forward error $\frac{\|x-\hat{x}\|_2}{\|x\|_2}$.
    (d) Verify the bound $\frac{||x-\hat{x}||}{||x||} \leq \kappa(A)\frac{||\hat{r}||}{||b||}$. (you may use numpy's LA.cond(A) command.)

16. TEXTBOOK EXERCISES: 1, 3, 4, 5.