1. In this question we will approximate the solution $x(t)$ of the IVP

$$x' = f(t,x) = -x^3 + \sin t$$
$$x(0) = 0.$$

Run the code for each of the methods: Euler, midpoint, and RK4 to approximate the graph of the solution $x(t)$ on the interval $[0, 10]$. This is a illustrative problem in math 328 because there is no way to find a formula for the solution in terms of known functions. The best one can do is approximate the solution as is done here. This is most often the case. It is rare indeed when an IVP can be solved for a formula as in a differential equation course.

2. In this question we will approximate $x(2)$ when $x(t)$ solves the IVP

$$x' = f(t,x) = x - t**2 + 1$$
$$x(0) = 0.5.$$

This is a baby problem in that we can find the exact solution by finding an integrating factor as you learned in your differential equation course. The exact solution is

$$x(t) = t^2 + 2t + 1 - \frac{1}{2}e^t.$$

Normally we would not approximate a solution to an IVP when we already have the exact solution (see problem 1 above). We do so in this case because we want to better understand the order of error in our methods.

(a) Use Euler's method with $h = 0.2$ to approximate $x(2)$ and then find the error by comparing with the exact solution $x(2)$. Repeat with $h = 0.02$. How do the errors compare with your understanding of the saying, "halve the stepsize, halve the error."?

(b) Use RK2 (= explicit trapezoid p.494.) method with $h = 0.2$ to approximate $x(2)$ and then find the error by comparing with the exact solution $x(2)$. Repeat with $h = 0.02$. How do the errors compare with your understanding of the saying, "halve the stepsize, quarter the error."?

(c) Use RK4 method with $h = 0.2$ to approximate $x(2)$ and then find the error by comparing with the exact solution $x(2)$. Repeat with $h = 0.02$. How do the errors compare with your understanding of the saying, "halve the stepsize, $\frac{1}{16}$th error."?

3. (final exam) In this problem we will approximate the integral $\int_1^3 \sqrt{1+x^3}dx$.

(a) Calculate the left endpoint Riemann sum approximation using $40, 400$, and 4000 equally-spaced subintervals. How many decimal places have stabilized?

(b) Approximate the integral using the trapezoid method with $40, 400$, and 4000 equally-spaced subintervals. How many decimal places have stabilized?

(c) Approximate the integral using Simpson's method with $40, 400$, and 4000 equally-spaced subintervals. How many decimal places have stabilized?

4. (final exam) Here is a cool problem. Repeat the previous problem by considering the integral $\int_1^3 \sqrt{1+x^3}dx$ from a differential equation viewpoint. Consider it as the solution $y(3)$ as the solution of the IVP $y' = \sqrt{1+x^3}, y(1) = 0$. Of course

$$y(t) = \int_1^t \sqrt{1+x^3}dx$$

by the fundamental theorem of calculus and thus $y(3) = \int_1^3 \sqrt{1+x^3}dx$

(a) Approximate the solution to the IVP using RK2 (trapezoid method) with stepsize $h = \frac{3-1}{4000}$. How does this compare with your approximation of the integral in the previous problem?

(b) Approximate the solution to the IVP using RK4 with stepsize $h = \frac{3-1}{4000}$. How does this compare with your approximation of the integral in the previous problem?

5. (textbook): CH $14 : 1, 4$, and CH $15 : 1, 3, 5, 7$.

6. (final exam): CH3 Exercise: 1, CH6 Example 6.2 on page $146 - 147$, CH7 Exercise: 3, CH8 Example 8.3 on page $224 - 225$,