Answer each question neatly on the line provided.

Name: ______________________ ID: ______________

1. (8 points) (Multiple Choice) Find the third row of R when $A = QR$ is the QR factorization $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

   A. $(1, 0, 0)$

   B. $(0, 0, 1)$

   C. $(1, 1, \frac{1}{2})$

   D. $(0, 0, \frac{1}{2})$

   E. none of these.

1. ______________

2. (8 points) Find an error bound after using the Trapezoid rule $\int_a^b \ln x \mathrm{d}x \approx \frac{b-a}{2}(\ln a + \ln b)$ to approximate $\int_1^2 \ln x dx$.

2. ______________

3. (5 points) [True/False] The following Python script uses Newton's method with initial guess $x_0 = 1$ to make 11 further approximations of $\sqrt[3]{7}$.

```
x = 1
for k in range(11):
    x = x - (x**3-7)/(3*x^2)
print(x)
```

3. ______________

4. (5 points) [True/False] Given the Cholesky factorization, $A = GG^T$, of a positive definite matrix A when G is lower triangular one can solve $Ax = b$ using the algorithmic idea: (1) evaluate $c = G^T b$, (2) solve $Gx = c$ for x by back substitution.

4. ______________

5. (5 points) [True or False] $L = \begin{bmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{bmatrix}$ for the $PA = LU$ factorization of the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$ by partial pivoting.

5. ______________

6. Let $g(x) = \frac{x+6}{3x-2}$

(a) (5 points) Find all fixed points of $g(x)$

(a) ______________

(b) (5 points) which fixed points of $g(x)$ from part (a) will FPI be guaranteed to converge when the initial guess is sufficiently.

(b) ______________

7. (8 points) Use the Bisection Method to find the first three approximate roots of $0 = x^2 - 2$ starting on the interval $[a, b] = [0, 2]$.

7. ______________

8. (10 points) Use the Taylor series with remainder to find the order of the error when using $\frac{-f(x+2h)+4f(x+h)-3f(x)}{2h}$ to approximate $f'(x)$.

8. ______________

9. (8 points) Around 1920 L. F. Richardson constructed a simple model to describe an "arms race" between two countries. If x and y are the annual military budgets of the two countries (in billions ofdollars), then the model $x' = -4x + 2y + 10$ and $y' = 5x - 4y + 2$ expresses the rates at which x and y change (in billions of dollars per year). Consider two countries when x=5 and y=6 this year (t=0). Use Euler's method with $\Delta t = 2$ years to estimate x(4), the military budget of country x after 4 years in billions of dollars.

9. ______________

10. (5 points) Find the least number of steps of the Bisection Method to guarantee the approximate root of $0 = x^2 - 2$ starting on the interval $[a, b] = [0, 4]$ is accurate within $\frac{1}{2^7}$.

10. ______________

11. (5 points) [True/False] An accurate way to describe the error in using Euler's method to approximate the solution to SIR model is, "if you $\frac{1}{2}$ the stepsize h then you $\frac{1}{4}$ the error."

11. ______________

12. The linear system

$$\begin{cases} 10x_1 + x_2 + x_3 = 12 \\ x_1 + 10x_2 + x_3 = 12 \\ x_1 + x_2 + 10x_3 = 12 \end{cases}. \tag{1}$$

has a unique solution $x_1 = x_2 = x_3 = 1$. Starting from $\vec{x}_0 = (0,0,0)^T$ find

(a) (5 points) one iteration using Jacobi and its error norm in $\mathcal{L}_1$

(a) ______________

(b) (5 points) one iteration using Gauss-Seidel and its error norm in $\mathcal{L}_1$

(b) ______________

13. (5 points) [True/False] If the condition number $\kappa(A) = .1$ and the relative forward error $\frac{\|x-\hat{x}\|}{\|x\|} = .003$ then relative backward error $\frac{\|\hat{r}\|}{\|b\|} \geq .03$ then the

13. ______________

14. (8 points) Use the infinity norm to find the relative residual $\frac{\|\hat{r}\|}{\|b\|}$ for the approximate solution $\hat{x} = \begin{bmatrix}1\\1\end{bmatrix}$ to the system $\begin{bmatrix}1 & 1\\3 & -4\end{bmatrix} x = \begin{bmatrix}3\\2\end{bmatrix}$.

14. ______________

15. (5 points) [True/False] $Q = \begin{bmatrix}1 & 1 & 1 & 1\\1 & -1 & -1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\end{bmatrix}$ is an orthogonal matrix.

15. ______________

1. (10 points) Use the composite Simpson's Rule

$$\int_a^b f(x)dx \approx \frac{h}{3}\left(f(a) + 2\sum_{k=1}^{\frac{r}{2}-1} f(t_{2k}) + 4\sum_{k=1}^{\frac{r}{2}} f(t_{2k-1}) + f(b)\right)$$

to approximate $\int_0^4 3\sqrt{x}dx$ when $r = 2, 20, 200$ subintervals. Compare your approximate answers to the exact answer by writing the exact error in each case. Then use stabilization of digits to describe how this comparison fits with the error formula for composite Simposon's rule.

2. (10 points) A sample of R grams of radium decays into lead at the rate $R' = -\frac{R}{233}$ grams per year. Using RK2 with step sizes $\Delta t = 5, .5, .05, .005$ years, make a table to estimate how much radium remains in a 0.72 gram sample after 40 years. Use stabilizing of digits to describe order of the method RK2.