4 Linear Algebra Background

Systems of Linear equations and algebraic eigenvalue (or singular value) problems arise frequently in large numerical computations, often as subproblems. Computational tasks associated with models arising in computer science, mathematics, statistics, the natural sciences, social sciences, various engineering disciplines, and business and economics all require the solution of such linear algebra systems.

4.1 Basics of Linear Systems

This section provides a quick review of linear algebra. You should be comfortable with the terms and ideas mentioned while reading the section. The two fundamental problems considered in the section are:

- Solve linear systems $Ax = b$ for the vector x when you are given a square matrix A and an input vector b.
- Solve the eigenvalue problem $Ax = \lambda x$ for the vector x and scalar λ when you are given a square matrix A.

4.1 Basics of Linear Systems

Find all solutions to the nonsingular system $\begin{cases} x_1 - 4x_2 = -10 \\ \frac{1}{2}x_1 - x_2 = -2 \end{cases}$

- Describe the Matrix Form, the Row Picture geometrically.
- the Column Picture geometrically. Do linear combinations "give" whole plane? What is the nullspace?

4.1 Basics of Linear Systems

Find all solutions to the singular system $\begin{cases} x_1 + x_2 = 5 \\ 3x_1 + 3x_2 = 16 \end{cases}$

- Describe the Matrix Form, the Row Picture geometrically.
- the Column Picture geometrically. Do linear combinations "give" whole plane? What is the nullspace?

4.1 Basics of Linear Systems

Find all solutions to the singular system $\begin{cases} x_1 + x_2 = 5 \\ 3x_1 + 3x_2 = 15 \end{cases}$

- Describe the Matrix Form, the Row Picture geometrically.
- the Column Picture geometrically. Do linear combinations "give" whole plane? What is the nullspace?

4.1 Basics of Linear Systems

Find all solutions to the "almost" singular system

$$\begin{cases} 1.0001x_1 + x_2 = 2.0001 \\ 3x_1 + 3x_2 = 6 \end{cases}$$

- $(1, 1)$ is the only solution. Yet, $(2, 0)$ is "almost" a solution and far from $(1, 1)$. This is no problem in linear algebra. It is a huge problem in numerical linear algebra.
- the Column Picture geometrically. Do linear combinations "give" whole plane? What is the nullspace?

4.1 Basics of Linear Systems

- $$\begin{cases} 2x - y = 0 \\ -x + 2y - z = -1 \\ -3y + 4z = 4 \end{cases}$$
- Matrix Form. Row Picture. Nullspace?
- Column Picture. Do linear combinations "give" whole plane?
- What happens in a system of equations with 9 equations and 9 unknowns $x_1, x_2, \ldots, x_9$.

4.1 Basics of Linear Systems

- Vector Space requirements.
- Addition and scalar multiples remain in space.
- Examples: lines?, planes?, quadrants?

4.1 Basics of Linear Systems

The following are equivalent for a square matrix A.

- A is nonsingular.
- $\det(A) \neq 0$.
- The columns of A are linearly independent.
- A^{-1} exists.
- range(A) $= \mathbb{R}^n$ and nullspace(A) $= \{0\}$.

4.1 Eigenvalues, Eigenvectors

The algebraic eigenvalue problem $Ax = \lambda x$ is fundamental. Basically one wants to characterize the action of matrix multiplication of a vector x by a square matrix Ax in simple terms, i.e as scalar multiplication λx. Together (λ, x) is an eigenpair of A and the set of all eigenvalues forms the spectrum of A.

4.1 Eigenvalues, Eigenvectors

- v is an eigenvector of a square matrix A if $Av = \lambda v$.
- Show that $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
- Show that $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
- Show that $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is not an eigenvector of $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

4.1 Finding Eigenvalues, Eigenvectors (by hand).

- $Av = \lambda v$ same as $(A - \lambda I)v = 0$.
- $(A - \lambda I)$ has a zero pivot.
- Solve $\det(A - \lambda I) = 0$, the *characteristic equation*.
- Solve $(A - \lambda I)v = 0$ for each λ.

4.1 Finding Eigenvalues, Eigenvectors (by hand).

- Example: Eigenvalues of projection matrices.
- Example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
- Example: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.
- Example (Rotation): $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

4.1 Spectral Decomposition

Let X be the matrix whose columns are eigenvectors of A, and suppose that X is square and nonsingular. Then

$$A = X\Lambda X^{-1}.$$

where Λ is the diagonal matrix of eigenvalues. Any matrix that can be decomposed in this fashion is called *diagonalizable*.

- Example: $A = \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix}$.
- Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4.1 Spectral Decomposition

Not all square matrices A are diagonalizable, i.e. there are square matrices that cannot be factored as

$$A = X\Lambda X^{-1}.$$

As numerical analysts we should ask ourselves is in which circumstances present potential difficulties when solving the eigenvalue problem? It turns out that slight perturbations of nondiagonalizable matrices are such a source of trouble.

4.1 Spectral Decomposition

EXAMPLE: consider the nondiagonalizable matrix $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ and then slightly perturb to get the diagonalizable matrix $\begin{bmatrix} 4 & 1.01 \\ .01 & 4 \end{bmatrix}$. Using MATLAB's *eig* function we find that the eigenvalues of the diagonalizable matrix are 4.1005 and 3.8995. This result is alarming because a perturbation of 0.001 in the matrix magnified a perturbation of 0.1005 in the eigenvalues.

4.2 Vector Norms

The material in this section is most likely new for you. You need to learn how to measure the size of a vector and the size of a matrix in order to be able to answer questions about how big is the error in a numerical linear algebra problem.

4.2 Vector Norms

In previous chapters we have defined relative and absolute errors for scalars (the absolute value) and have used them to assess the quality of an algorithm. Errors measure the distance between the true solution x and the computed solution $\hat{x}$ from your algorithm. Since solutions to linear algebra problems will be vectors and (sometimes) matrices, you need to be able to measure the size of vectors and matrices. Unfortunately there are many ways to measure the size of a vector and a matrix–some measures are more suitable for some problems; other measures are more suitable for different problems. We call each such measure a *norm* which is a generalization of the absolute value.

4.2 Vector Norms

Given the vector $\vec{x} = (x_1, x_2, \ldots, x_n)$. Here are the only vector norms you need to memorize. You've already used the second one in calculus III. It is based on the Pythagorean theorem.

- $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.
- $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\vec{x}^T \vec{x}}$.
- $\|x\|_\infty = \max |x_i|$.

4.2 Vector Norms Examples

Measure the distance between the vectors $\vec{x} = \begin{bmatrix} 11 \\ 12 \\ 13 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 12 \\ 14 \\ 16 \end{bmatrix}$ in each norm.

4.2 Matrix Norms Examples

Let $A = \begin{bmatrix} 1 & 3 & 7 \\ -4 & 1.2725 & -2 \end{bmatrix}$. Here are the only matrix norms you need for this class.

- $\|A\|_1 = \max(5, 4.2725, 9) = 9$.
- $\|x\|_\infty = \max(11, 7.2725) = 11$.
- $\|A\|_2 = \max_{\|x\|_2=1} \sqrt{(A\vec{x})^T A\vec{x}}$.
- $\|A\|_2 = \sqrt{\rho(A^T A)}$ when $\rho(B) =$ largest eigenvalue of B in absolute value.

4.3 Special Classes of Matrices

These special types of square matrices arise often in science, engineering, and numerical linear algebra.

- Symmetric: $A^T = A$.
- Positive Definite: $x^T A x > 0$ for all vectors $x \neq 0$.
- Orthogonal: A is square and $A^T A = I$.

4.3 Examples: Special Classes of Matrices

Determine if each of the following matrices are symmetric, positive definite, and (or) orthogonal.

- $A = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix}$.
- $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$.
- $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

4.3 Spectral Theorem

A matrix A is orthogonally diagonalizable, $A = Q\Lambda Q^T$, if and only if A is symmetric. If A happens to be symmetric and positive definite, all eigenvalues λ_i are positive.

4.4 Singular Value Decomposition

Given a rectangular $m \times n$ rank r matrix A, the SVD gives

$$A = U\Sigma V^T$$

when U is $m \times m$ orthogonal, V is $n \times n$ orthogonal, and Σ is $m \times n$ diagonal with diagonals $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$.

4.4 Example Singular Value Decomposition

Find the SVD of $A = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}$.

- $A^T A = \begin{bmatrix} 85 & -30 \\ -30 & 40 \end{bmatrix}$.
- $\lambda_1 = 100, \vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.
- $\lambda_2 = 100, \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Ch 4: Examples

Represent the solution to the system

$$2x - 5y = 8$$
$$3x + 9y = -12$$

as the intersection of two lines.

Ch 4: Examples

Rewrite the system

$$\begin{aligned} x + 2y &= 7 \\ 3x + y &= 11 \end{aligned}$$

as the linear combination of the two columns. Then represent this linear combination geomtrically in the plane and draw the solution using the parallelogram law.

Ch 4: Examples

Instead of using the geometry of linear combinations to solve the system

$$\begin{aligned} x + 2y &= 7 \\ 3x + y &= 11, \end{aligned}$$

solve it by factoring $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ as $A = LU$ and then using forward and then back substituion.

Ch 4: Examples

- Find $\|x\|_1, \|x\|_2$, and $\|x\|_\infty$ when $x = \begin{bmatrix} 2.1 \\ -3 \\ 1. -5 \end{bmatrix}$.
- Find $\|A\|_1, \|A\|_2$, and $\|A\|_\infty$ when $A = \begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & -3 \\ 1 & -7 & 0 \end{bmatrix}$.

Ch 4: Examples

Find all the eigenvalues of each of the following square matrices A. Then find a basis for each eigenspace and diagonalize A by factoring $A = X\Lambda X^{-1}$, if possible.

- $A = \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$
- $A = \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix}$
- $A = \begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix}$

Ch 4: Examples

- Decompose $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ as $A = X\Lambda X^{-1}$.
- Decompose the symmetric matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ as $A = Q\Lambda Q^T$
- Find the SVD of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.