Math 39100 K (23535) - Homework Solutions 01

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Variables Separable

You will need to remember:

$$\frac{d\tan(x)}{dx} = \sec^2(x) \quad \text{and} \quad \frac{d\sec(x)}{dx} = \sec(x)\tan(x).$$

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \qquad \text{and}$$

$$\int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C.$$

$$\cos(2x) = \cos^2(x) - sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x).$$

So: $\cos^2(x) = \frac{1}{2}[1+\cos(2x)]$, and $\sin^2(x) = \frac{1}{2}[1-\cos(2x)]$.

Also remember:

$$\int \frac{x}{1+x^2}\,dx \;=\; \frac{1}{2}\ln(1+x^2)+C$$

by u-substitution, but

$$\int \frac{1}{1+x^2}\,dx \;=\; \arctan(x)+C.$$

Variables Separable

2.2/5. $\int \sec^2(2y)dy \;=\; \int \cos^2(x)dx.$

$\frac{1}{2}\tan(2y) \;=\; \frac{1}{4}\sin(2x) + \frac{1}{2}x + C.$

2.2/11. $\int_1^y \; a \; da \;=\; -\int_0^x \; be^b \; db.$ Use Integration by Parts.

$\frac{1}{2}[y^2 - 1] = -1 + e^x - xe^x.$ $y = +\sqrt{2e^x - 2xe^x - 1}.$

Plus sign from initial condition.

Homogeneous

2.2/33. $x\frac{dz}{dx} + z = \frac{4z-3}{2-z}$.

$$x\frac{dz}{dx} = \frac{z^2 + 2z - 3}{2 - z} = \frac{(z-1)(z+3)}{2-z}.$$

$$\frac{2-z}{(z-1)(z+3)} = \frac{A}{(z-1)} + \frac{B}{(z+3)}.$$

$$2 - z = A(z+3) + B(z-1).$$

Substitute $z = -3, 1$ to get

$$B = -\frac{5}{4},\ A = \frac{1}{4}.$$

$$\ln|x| + C = -\frac{5}{4}\ln|z-1| + \frac{1}{4}\ln|z+3|.$$

$$\ln|x| + C = -\frac{5}{4}\ln|y-x| + \frac{1}{4}\ln|y+3x| + \frac{5}{4}\ln|x| - \frac{1}{4}\ln|x|.$$

$$C = \frac{|y+3x|}{|y-x|^5}.$$

Linear Equations

2.1/6 Make the coefficient of y' equal to 1:

$$\frac{dy}{dt} + (\frac{2}{t})y = \frac{\sin(t)}{t}.$$

$$\mu = \exp[\int \frac{2}{t} dt] = \exp[2\ln(t)] = t^2.$$

$$t^2 \frac{dy}{dt} + 2ty \;=\; [t^2 y]' \;=\; t\sin(t).$$

$$t^2 y = C - t\cos(t) + \sin(t).$$

Exact Equations

2.6/9 $Mdx + Ndy =$
$(ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x)dx \; + \; (xe^{xy}\cos(2x) - 3)dy.$

$$\frac{\partial M}{dy} = \frac{\partial N}{dx} = e^{xy}\cos(2x) + xye^{xy}\cos(2x) - 2xe^{xy}\sin(2x).$$

$$F = \int (xe^{xy}\cos(2x) - 3)\; dy = e^{xy}\cos(2x) - 3y + H(x).$$

$$\begin{aligned}\frac{\partial F}{dx} &= ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + H'(x) \\ &= ye^{xy}\cos(2x) - 2e^{xy}\sin(2x) + 2x.\end{aligned}$$

$H(x) = x^2$ and so $\quad e^{xy}\cos(2x) - 3y + x^2 = C.$

Tank Problems

2.3/2. $V_0 = 120L, S_0 = 0$. Input and Output $2L/min$.
Concentration of Input $\gamma g/L$.

In L/min: $\frac{dV}{dt} = 2 - 2 = 0$. So $V = V_0 = 120$.
In g/min: $\frac{dS}{dt} = 2 \cdot \gamma - 2\frac{S}{V} = \frac{120\gamma - S}{60}$.

$$\frac{dS}{S - 120\gamma} = -\frac{dt}{60}; \qquad S - 120\gamma = Cexp[-t/60].$$

$$C = -120\gamma; \qquad S = 120\gamma(1 - exp[-t/60]).$$

As $t \to \infty$ S tends to 120γ.

2.3/4. $V_0 = 200gal, S_0 = 100lb$. Input $3gal/min$, Output $2gal/min$. Concentration of Input $1lb/gal$.

In gal/min: $\frac{dV}{dt} = 3 - 2 = 1$. So $V = V_0 + t = 200 + t$.
In lb/min: $\frac{dS}{dt} = 3 \cdot 1 - 2\frac{S}{V} = 3 - \frac{2S}{200+t}$.

$$\frac{dS}{dt} + \frac{2}{200+t}S = 3; \qquad \mu = (200+t)^2.$$

$$[(200+t)^2 S]' = 3(200+t)^2; \qquad (200+t)^2 S = (200+t)^3 + C;$$

So $C \ = \ 200^2 100 - 200^3 \ = \ -200^2 100$.

$$S = (200+t) - 100\left(\frac{200}{(200+t)}\right)^2; \qquad S/V = 1 - (1/2)\left(\frac{200}{(200+t)}\right)^3.$$

Interest Problems

2.3/8. Initial Amount $S_0 = 0$. Interest Rate $r\%/yr$. Input $k\$/yr$.

$$\frac{dS}{dt} = rS + k; \qquad \frac{dS}{S + (k/r)} = rdt.$$

$$S + (k/r) = Ce^{rt}; \qquad C = (k/r).$$

$$S = (k/r)(e^{rt} - 1)$$

(b) $1,000,000 = (k/.075)(e^{.075 \cdot 40} - 1)$.
(c) $1,000,000 = (4000/r)(e^{40r} - 1)$. Can't solve explicitly.

2.3/9. Initial Amount $S_0 = 8000$. Interest Rate $.1\%/yr$. Output $k\$/yr$. $S_3 = 0$.

$$\frac{dS}{dt} = .1S - k; \qquad \frac{dS}{S - 10k} = .1dt.$$

$$S - 10k = Ce^{.1t}; \qquad 8000 - 10k = C.$$

$$S = 8000e^{.1t} - 10k(e^{.1t} - 1); \qquad k = \frac{800e^{.3}}{e^{.3} - 1} = 3087.$$

Interest paid $= 3k - 8000 = 1261$.

Reduction of Order

2.9/45 $2y^2 v \frac{dv}{dy} + 2yv^2 = 1,$ or
$(2y^2 v)dv + (2yv^2 - 1)dy = 0.$
Exact: $y^2 v^2 - y = C_1$ and so $\frac{dy}{dx} = v = \pm\frac{\sqrt{C_1+y}}{y}.$
Let $u = C_1 + y$

$$\pm x + C_2 = \int dx = \int \frac{u - C_1}{\sqrt{u}}\, du = \frac{2}{3}u^{3/2} - 2C_1 u^{1/2}.$$

$$\pm x + C_2 = \sqrt{C_1 + y}[\frac{2}{3}(C_1 + y) - 2C_1] = \sqrt{C_1 + y}[\frac{2}{3}(y - 2C_1)].$$

Miscellaneous Problems

1. Linear: Rewrite as: $\frac{dy}{dx} + \frac{2}{x}y = x^2$.

$$\mu = exp[\int \frac{2}{x}dx] = exp[2\ln(x)] = x^2.$$

$$[x^2y]' = x^2\frac{dy}{dx} + 2xy = x^4.$$

$$x^2y = \frac{x^5}{5} + C; \quad y = \frac{x^3}{5} + Cx^{-2}.$$

2. Variables separable: $2y + \cos y = x + \sin x + C$.

3. Not separable, linear, or homogeneous. Exact:
Rewrite as: $(3+3y^2-x)dy-(2x+y)dx=0$.

$$\frac{\partial}{\partial x}(3+3y^2-x)=-1=\frac{\partial}{\partial y}-(2x+y).$$

$$F(x,y)=\int(3+3y^2-x)dy=3y+y^3-xy+H(x).$$

$$\frac{\partial F}{\partial x}=-y+H'(x)=-(2x+y).$$

$$H'(x)=-2x;\quad H(x)=-x^2.$$

$$F(x,y)=3y+y^3-xy-x^2;\ \text{Solution: } 3y+y^3-xy-x^2=C.$$

$$y(0)=0;\quad \text{and so } C=0.$$

4. Variables Separable:
$\frac{dy}{dx} = 3(1-2x) + y(1-2x) = (3+y)(1-2x)$.

$$\int \frac{dy}{3+y} = \int (1-2x)dx; \quad \ln(3+y) = x - x^2 + C.$$

$$y = -3 + Cexp[x - x^2].$$

Linear: $\frac{dy}{dx} + (2x-1)y = 3 - 6x$.

$$\mu = exp[-x+x^2]; \; exp[-x+x^2]y = \int (3-6x)exp[-x+x^2]dx.$$

$$u - sub : u = -x + x^2 : \; exp[-x+x^2]y = -3exp[-x+x^2] + C.$$

5. Exact: $(x^2 + 2xy)dy \; + \; (2xy + y^2 + 1)dx \; = \; 0.$

$$\frac{\partial}{\partial x}(x^2 + 2xy) = 2x + 2y = \frac{\partial}{\partial y}(2xy + y^2 + 1).$$

$$F(x, y) = \int (x^2 + 2xy)dy = x^2y + xy^2 + H(x).$$

$$\frac{\partial F}{\partial x} = 2xy + y^2 + H'(x) = (2xy + y^2 + 1).$$

$$H'(x) = 1; \; H(x) = x; \; F(x, y) = 2xy + y^2 + x$$

$$2xy + y^2 + x = C.$$

6. Linear, Combine y terms: $\frac{dy}{dx} + (1 + \frac{1}{x})y = \frac{1}{x}$.

$$\mu = exp[\int (1 + \frac{1}{x})dx] = xe^x.$$

$$[xe^x y]' = xe^x \frac{dy}{dx} + (x+1)e^x y = e^x.$$

$$xe^x y = e^x + C; \quad y(1) = 0; \quad 0 = e + C; \quad C = -e.$$

$$y = (1 - e^{(1-x)})/x.$$

7. Variables Separable; 8. Linear; 9. Exact

10. Variables Separable: $(x^2 + x - 1)y + x^2(y - 2)\frac{dy}{dx} = 0.$

$$\frac{y-2}{y}dy = -\frac{x^2 + x - 1}{x^2}dx.$$

$$y - 2\ln|y| = C - x - \ln|x| - x^{-1}.$$

11. Exact; 12. Linear;

13. Variables Separable: $\frac{dy}{dx} = (1 + 2x)(1 + y^2).$

$$arctan(y) = x + x^2 + C; \quad y = \tan(x + x^2 + C).$$

14. Exact: $F(x,y) = xy + \frac{x^2}{2} + y^2$.

$$xy + \frac{x^2}{2} + y^2 = C; \;\; y(2) = 3, \;\; 6+2+9 = C; \;\; xy + \frac{x^2}{2} + y^2 = 17.$$

Homogeneous: $\frac{dy}{dx} = -\frac{x+y}{x+2y}$; Let $z = y/x$.

$$\frac{dz}{dx} = -z - \frac{1+z}{1+2z} = -\frac{2z^2+2z+1}{1+2z}.$$

$$\int \frac{1+2z}{2z^2+2z+1} dz = -\int \frac{dx}{x}.$$

$$u - sub : \frac{1}{2} \ln |2z^2 + 2z + 1| = -\ln x + C;$$

$$2(y/x)^2 + 2(y/x) + 1 = Cx^{-2}; \;\; 2y^2 + 2xy + x^2 = C; \;\; C = 34.$$

28. Linear: $\frac{dy}{dx} + \frac{2}{x}y = -3.$

$$\mu = x^2; \quad [x^2y]' = -3x^2; \quad x^2y = -x^3 + C; \quad y = -x + Cx^{-2}.$$

Homogeneous: $\frac{dy}{dx} = -2(y/x) - 3;$ Let $z = y/x$.

$$\frac{dz}{dx} = -3 - 3z; \quad \frac{dz}{1+z} = -3\frac{dx}{x}.$$

$$\ln|1+z| = -3\ln|x| + C; \quad 1 + (y/x) = Cx^{-3}; \quad x^3 + x^2y = C.$$

29. Homogeneous (not Exact): $x\frac{dz}{dx} + z = \frac{1+z}{1-z}$.

$$\int \frac{1-z}{1+z^2} dz = \int \frac{dx}{x}.$$

$$\arctan(z) - \frac{1}{2}\ln(1+z^2) = \ln(x) + C = \frac{1}{2}\ln(x^2) + C.$$

$$\arctan(y/x) - \frac{1}{2}\ln(x^2+y^2) = C.$$

Example $y'' - y = 0$ via Reduction of Order:

$$v\frac{dv}{dy} = y, \qquad v^2 = y^2 + C_1, \qquad \frac{dy}{dx} = v = \pm\sqrt{y^2 + C_1}.$$

By Trig Substitution
$\pm x + C_2 = \ln|y + \sqrt{y^2 + C_1}|, \qquad \sqrt{y^2 + C_1} = C_2 e^{\pm x} - y.$

Square to get $C_1 = C_2^2 e^{\pm 2x} - 2C_2 y e^{\pm x}$. Finally,

$$y = C_1 e^x + C_2 e^{-x}.$$

Instead, use $y = e^{rx}$ as test function. Characteristic Equation $r^2 - 1 = 0$ with roots $r = \pm 1$. So directly,

$$y = C_1 e^x + C_2 e^{-x}.$$