Every real number is the radian measure of some angle

Every real number, viewed as a number of radians, represents an angle. The initial side of the angle is a horizontal arrow pointing right from the origin. Rotate that arrow counterclockwise (or clockwise) with each full turn counting as $2\pi$ (or $-2\pi$) radians. The final position of the arrow is the terminal side of the angle.

In the example at the right, the initial side is rotated through $\frac{3}{4}$ of a complete circle. The rotation is suggested by the red dashed arc. The angle $\theta$ corresponds to $\frac{3}{4}$ of a complete trip around the circle, and so $\theta = \frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$ rad = $270^\circ$.

More examples are on the next slide.
### Radian Measure

<table>
<thead>
<tr>
<th>Radians</th>
<th>Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2\pi$</td>
<td>360</td>
</tr>
<tr>
<td>$\pi$</td>
<td>180</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>90</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>60</td>
</tr>
<tr>
<td>$\pi/4$</td>
<td>45</td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>30</td>
</tr>
</tbody>
</table>

- $\frac{3\pi}{4}\text{ rad} = 135^\circ$
- $\pi\text{ rad} = 180^\circ$
- $\frac{11\pi}{6}\text{ rad} = 330^\circ$
- $\frac{5\pi}{4}\text{ rad} = 225^\circ$
- $-\frac{5\pi}{4}\text{ rad} = -225^\circ$
- $-\pi\text{ rad} = -180^\circ$
- $-\frac{\pi}{6}\text{ rad} = -30^\circ$
- $-\frac{3\pi}{4}\text{ rad} = -135^\circ$
Trig functions of angles in Quadrant 1

Let \( \theta \) be any angle. Place its initial side on the positive x-axis. Suppose the arrow tip of the terminal side of \( \theta \) is at point \( P(x, y) \) in the x,y-plane. Let \( r = \sqrt{x^2 + y^2} \) be the distance from \( P(x, y) \) to the origin. \( r \) is positive!

**The trig functions of angle \( \theta \)**

\[
\begin{align*}
\cos(\theta) &= \frac{x}{r} & \sec(\theta) &= \frac{1}{\cos(\theta)} = \frac{r}{x} \\
\sin(\theta) &= \frac{y}{r} & \csc(\theta) &= \frac{1}{\sin(\theta)} = \frac{r}{y} \\
\tan(\theta) &= \frac{y}{x} & \cot(\theta) &= \frac{1}{\tan(\theta)} = \frac{x}{y}
\end{align*}
\]

In the picture at the right, the arrow tip of the terminal side is at \( P(4,3) \). Thus
\[
\begin{align*}
x &= 4, y &= 3, r &= \sqrt{x^2 + y^2} = 5 \text{ and so} \\
\cos(\theta) &= \frac{x}{r} = \frac{4}{5} & \sec(\theta) &= \frac{r}{x} = \frac{5}{4} \\
\sin(\theta) &= \frac{y}{r} = \frac{3}{5} & \csc(\theta) &= \frac{r}{y} = \frac{5}{3} \\
\tan(\theta) &= \frac{y}{x} = \frac{3}{4} & \cot(\theta) &= \frac{r}{y} = \frac{4}{3}
\end{align*}
\]
Trig functions of angles in Quadrants 2,3,4

On this slide we are concerned not with the values of trig functions, but with their signs. Remember that $r$ is always positive, while $x$ and $y$ can be positive or negative.

- $\cos(\theta) = x/r$ has the same sign as $x$, which is positive in Q1 and Q4, negative in Q2 and Q3.
- $\sin(\theta) = y/r$ has the same sign as $y$, which is positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$ is positive in Q1 and Q3, negative in Q2 and Q4.
On this slide we are concerned not with the values of trig functions, but with their signs. Remember that $r$ is always positive, while $x$ and $y$ can be positive or negative.

- $\cos(\theta) = \frac{x}{r}$ has the same sign as $x$, which is positive in Q1 and Q4, negative in Q2 and Q3.
- $\sin(\theta) = \frac{y}{r}$ has the same sign as $y$, which is positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = \frac{y}{x}$ is positive in Q1 and Q3, negative in Q2 and Q4.

You may choose to derive these facts from ASTC, but it's better to think of the picture. The quadrant by quadrant results are:

- **Quadrant 1**: All: cos, sin, tan are +.
- **Quadrant 2**: sin is +, cos and tan are −.
- **Quadrant 3**: tan is +, cos and sin are −.
- **Quadrant 4**: cos is +, sin and tan are −.
On this slide we are concerned not with the values of trig functions, but with their signs. Remember that $r$ is always positive, while $x$ and $y$ can be positive or negative.

- $\cos(\theta) = \frac{x}{r}$ has the same sign as $x$, which is positive in Q1 and Q4, negative in Q2 and Q3.
- $\sin(\theta) = \frac{y}{r}$ has the same sign as $y$, which is positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = \frac{y}{x}$ is positive in Q1 and Q3, negative in Q2 and Q4.

You may choose to derive these facts from ASTC, but it's better to think of the picture.

The quadrant by quadrant results are:

- **Quadrant 1**: All: cos, sin, tan are $+$.
- **Quadrant 2**: Sin is $+$, cos and tan are $-$.
On this slide we are concerned not with the values of trig functions, but with their signs. Remember that $r$ is always positive, while $x$ and $y$ can be positive or negative.

- $\cos(\theta) = x/r$ has the same sign as $x$, which is positive in Q1 and Q4, negative in Q2 and Q3.
- $\sin(\theta) = y/r$ has the same sign as $y$, which is positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$ is positive in Q1 and Q3, negative in Q2 and Q4.

You may choose to derive these facts from ASTC, but it’s better to think of the picture.

The quadrant by quadrant results are:
- Quadrant 1: All: cos, sin, tan are +.
- Quadrant 2: Sin is +, cos and tan are −.
- Quadrant 3: Tan is +, cos and sin are −.
- Quadrant 4: Cos is +, sin and tan are −.
Trig functions of angles in Quadrants 2, 3, 4

On this slide we are concerned not with the values of trig functions, but with their signs. Remember that $r$ is always positive, while $x$ and $y$ can be positive or negative.

- $\cos(\theta) = x/r$ has the same sign as $x$, which is positive in Q1 and Q4, negative in Q2 and Q3.
- $\sin(\theta) = y/r$ has the same sign as $y$, which is positive in Q1 and Q2, negative in Q3 and Q4.
- $\tan(\theta) = y/x$ is positive in Q1 and Q3, negative in Q2 and Q4.

You may choose to derive these facts from ASTC, but it's better to think of the picture.

The quadrant by quadrant results are:
- Quadrant 1: All: cos, sin, tan are +.
- Quadrant 2: Sin is +, cos and tan are −.
- Quadrant 3: Tan is +, cos and sin are −.
- Quadrant 4: Cos is +, sin and tan are −.
We are now going to calculate trig functions of closely related angles. Start with angle $\theta$ from before. It will reappear in the next 3 pictures.

\[
\cos(\theta) = \frac{x}{r} = \frac{4}{5} \\
\sin(\theta) = \frac{y}{r} = \frac{3}{5} \\
\tan(\theta) = \frac{y}{x} = \frac{4}{3}
\]

In this example, $\theta$ is a Quadrant 1 angle. This means: $\theta$ is an angle whose terminal side lies in Quadrant 1.
All trig functions of $\theta$ are positive. $\theta$ will reappear in the next 3 slides as the reference angle of angles in Quadrants 2, 3, and 4.

**Definition**

The reference angle of a given angle is the acute ($< 90^\circ$) angle between the $x$-axis and the given angle’s terminal line.

The reference angle of angle $\theta$ is colored yellow on these slides. It is a positive angle, between 0 and 90°. All of its trig functions are positive.
The Quadrant 2 angle $\theta_2$ shown below has *reference angle* $\theta = 180^\circ - \theta_2$. The endpoint of the terminal side of $\theta_2$ is $P_2(-4, 3)$ Thus $x = -4, y = 3, r = \sqrt{(-4)^2 + 3^2} = 5$.

$$\cos(\theta_2) = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5}$$
$$\sin(\theta_2) = \frac{y}{r} = \frac{3}{5}$$
$$\tan(\theta_2) = \frac{y}{x} = \frac{3}{-4} = -\frac{3}{4}$$

Since $\theta_2$ is a Quadrant 2 (Q2) angle, $\cos(\theta_2)$ is negative since $x$ is positive in Q2. $\sin(\theta_2)$ is positive since $y$ is positive in Q2. $\tan(\theta_2)$ is negative since $y/x$ is negative in Q2.

It's easy to compute the trig functions of $\theta_2$ if you already know the trig functions of its reference angle $\theta$. For $\theta_2$ in Quadrant 2:

$$\cos(\theta_2) = -\cos(\theta).$$
$$\sin(\theta_2) = + \sin(\theta).$$
$$\tan(\theta_2) = - \tan(\theta).$$
The Quadrant 3 angle \( \theta_3 \) shown below has reference angle \( \theta_3 - 180^\circ \). The endpoint of the terminal side of \( \theta_3 \) is \( P_3(-4, -3) \). Thus \( x = -4, y = -3, r = \sqrt{(-4)^2 + (-3)^2} = 5 \).

\[
\begin{align*}
\cos(\theta_3) & = \frac{x}{r} = \frac{-4}{5} = -\frac{4}{5} \\
\sin(\theta_3) & = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5} \\
\tan(\theta_3) & = \frac{y}{x} = \frac{-3}{-4} = \frac{3}{4} 
\end{align*}
\]

Since \( \theta_3 \) is a Quadrant 3 (Q3) angle, 
\( \cos(\theta_3) \) is negative since \( x \) is positive in Q3. 
\( \sin(\theta_3) \) is negative since \( y \) is negative in Q3. 
\( \tan(\theta_3) \) is positive since \( y/x \) is positive in Q3.

It’s easy to compute the trig functions of \( \theta_3 \) if you already know the trig functions of its reference angle \( \theta \). For \( \theta_3 \) in Quadrant 3:
\[
\begin{align*}
\cos(\theta_3) & = -\cos(\theta) \\
\sin(\theta_3) & = -\sin(\theta); \\
\tan(\theta_3) & = +\tan(\theta).
\end{align*}
\]
The Quadrant 4 angle $\theta_4$ shown below has reference angle $\theta = 360^\circ - \theta_4$. The endpoint of the terminal side of $\theta_4$ is $P_4(4, -3)$ Thus $x = 4, y = -3, r = \sqrt{4^2 + (-3)^2} = 5$.

\[
\cos(\theta_4) = \frac{x}{r} = \frac{4}{5}
\]
\[
\sin(\theta_4) = \frac{y}{r} = \frac{-3}{5} = -\frac{3}{5}
\]
\[
\tan(\theta_4) = \frac{y}{x} = \frac{-3}{4} = -\frac{3}{4}
\]

Since $\theta_4$ is a Quadrant 4 (Q4) angle, $\cos(\theta_4)$ is positive since $x$ is positive in Q4. $\sin(\theta_4)$ is negative since $y$ is negative in Q4. $\tan(\theta_4)$ is negative since $y/x$ is negative in Q4.

It’s easy to compute the trig functions of $\theta_4$ if you already know the trig functions of its reference angle $\theta$. For $\theta_4$ in Quadrant 4:

\[
\cos(\theta_4) = +\cos(\theta)
\]
\[
\sin(\theta_4) = -\sin(\theta);
\]
\[
\tan(\theta_4) = -\tan(\theta).
\]
Finding the reference angle of a given angle.

We abbreviate "the reference angle of $\theta_2$" by $\text{Ref}(\theta_2)$. It is the angle $\theta$ shown in yellow on the previous slides. $\text{Ref}(\theta_2)$ is the smaller angle between the terminal line of $\theta_2$ and the x-axis.

The reference angle is always positive: $0^\circ < \text{Ref}(\theta_2) < 90^\circ$

Similar statements hold for the angles $\theta_3$ and $\theta_4$.

**Summary: How to find the reference angle of an angle given in degrees**

- **Quadrant 1:** If $0^\circ < \theta_1 < 90^\circ$ then $\text{Ref}(\theta_1) = \theta_1$.
- **Quadrant 2:** If $90^\circ < \theta_2 < 180^\circ$ then $\text{Ref}(\theta_2) = 180^\circ - \theta_2$.
- **Quadrant 3:** If $180^\circ < \theta_3 < 270^\circ$ then $\text{Ref}(\theta_3) = \theta_3 - 180^\circ$.
- **Quadrant 4:** If $270^\circ < \theta_4 < 360^\circ$ then $\text{Ref}(\theta_4) = 360^\circ - \theta_4$.
- **Angles** $0^\circ, \pm 90^\circ, \pm 180^\circ$... *that are a multiple of 90° have undefined reference angle.*
- *If $\theta < 0$ or $\theta > 360^\circ$, add or subtract a multiple of 360° to/from $\theta$ to get an angle $\theta'$ with $0^\circ \leq \theta' < 360^\circ$. Then $\text{Ref}(\theta) = \text{Ref}(\theta')$, computed according to the above rules.*
Summary: How to find the reference angle of an angle given in radians

- **Quadrant 1:** If $0 < \theta_1 < \frac{\pi}{2}$ then $\text{Ref}(\theta_1) = \theta_1$.
- **Quadrant 2:** If $\frac{\pi}{2} < \theta_2 < \pi$ then $\text{Ref}(\theta_2) = \pi - \theta_2$.
- **Quadrant 3:** If $\pi < \theta_3 < \frac{3\pi}{2}$ then $\text{Ref}(\theta_3) = \theta_3 - \pi$.
- **Quadrant 4:** If $\frac{3\pi}{2} < \theta_4 < 2\pi$ then $\text{Ref}(\theta_4) = 2\pi - \theta_4$.

Angles $0, \pm \frac{\pi}{2}, \pm \pi, \pm \frac{3\pi}{2}, \ldots$ that are a multiple of $\frac{\pi}{2}$ have undefined reference angle.

- If $\theta < 0$ or $\theta > 2\pi$, add or subtract a multiple of $2\pi$ to/from $\theta$ to get an angle $\theta'$ with $0 \leq \theta' < 2\pi$. Then $\text{Ref}(\theta) = \text{Ref}(\theta')$, computed according to the above rules.

**Example 1:** Find the cosines of the following angles: $150^\circ, 225^\circ, 300^\circ$

**Solution:**
- **150°:** Terminal line in Q2: $\text{Ref}(150^\circ) = 180^\circ - 150^\circ = 30^\circ$.
  Since cosine is negative in Q2, $\cos(150^\circ) = -\cos(30^\circ) = -\frac{\sqrt{3}}{2}$
- **225°:** Terminal line in Q3: $\text{Ref}(225^\circ) = 225^\circ - 180^\circ = 45^\circ$.
  Since cosine is negative in Q3, $\cos(225^\circ) = -\cos(45^\circ) = -\frac{1}{\sqrt{2}}$
- **300°:** Terminal line in Q4: $\text{Ref}(300^\circ) = 360^\circ - 300^\circ = 60^\circ$.
  Since cosine is positive in Q4, $\cos(300^\circ) = +\cos(60^\circ) = \frac{1}{2}$
The following summary need not be memorized if you understand the previous slides. Given any angle $\theta$, let $\overline{\theta}$ be the reference angle of $\theta$.

### Trig functions of angles in terms of the reference angle

- $\cos(\theta) = \cos(\overline{\theta})$ for angles $\theta$ ending in Q1 and Q4.
- $\cos(\theta) = -\cos(\overline{\theta})$ for angles $\theta$ ending in Q2 and Q3.
- $\sin(\theta) = \sin(\overline{\theta})$ for angles $\theta$ ending in Q1 and Q2.
- $\sin(\theta) = -\sin(\overline{\theta})$ for angles $\theta$ ending in Q3 and Q4.
- $\tan(\theta) = \tan(\overline{\theta})$ for angles $\theta$ ending in Q1 and Q3.
- $\tan(\theta) = -\tan(\overline{\theta})$ for angles $\theta$ ending in Q2 and Q4.

**Example 2:** Find the reference angle of $920^\circ$.

**Solution:** $920^\circ - 360^\circ - 360^\circ = 200^\circ$. Since $180^\circ < 200^\circ < 270^\circ$, angle $200^\circ$ is in Q3. Its reference angle is $200^\circ - 180^\circ = 20^\circ$.

**Example 3:** Find the reference angle of $\frac{19\pi}{3}$.

**Solution:** Method 1: Convert to degrees: $19\left(\frac{\pi}{3}\right) = 19(60^\circ) = 1140^\circ$.

Then $1140^\circ - 360^\circ - 360^\circ - 360^\circ = 60^\circ$ is in Q1, and so its reference angle is $60^\circ$.

Method 2: Keep radians. $\frac{19\pi}{3} - 2\pi - 2\pi - 2\pi = \frac{19\pi}{3} - 6\pi = \frac{19\pi}{3} - \frac{18\pi}{3} = \frac{\pi}{3}$ is in Q1 and so its reference angle is $\frac{\pi}{3}$. 

Stanley Ocken  M19500 Precalculus Chapter 6.3 Trigonometric functions of angles
The Pythagorean trigonometric identities

If the terminal side of angle $\theta$ goes from the origin to $P(x, y)$

- The distance from $P$ to the origin is $r = \sqrt{x^2 + y^2}$, and $x^2 + y^2 = r^2$.

- $\cos(\theta) = \frac{x}{r}$
- $\sin(\theta) = \frac{y}{r}$
- $\tan(\theta) = \frac{y}{x}$
- $\sec(\theta) = \frac{r}{x}$
- $\csc(\theta) = \frac{r}{y}$
- $\cot(\theta) = \frac{x}{y}$

In what follows, recall that powers of trig functions are abbreviated somewhat strangely:

$(\cos(\theta))^2$ is written $\cos^2(\theta)$, for instance.

Divide $x^2 + y^2 = r^2$ by $r^2$ to get

$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ and so

$(\cos(\theta))^2 + (\sin(\theta))^2 = 1$. This is usually written as $\cos^2(\theta) + \sin^2(\theta) = 1$.

Similarly, dividing $x^2 + y^2 = r^2$ by $x^2$ gives the identity $1 + \tan^2(\theta) = \sec^2(\theta)$.

Similarly, dividing $x^2 + y^2 = r^2$ by $y^2$ gives the identity $1 + \cot^2(\theta) = \csc^2(\theta)$.

These identities should be memorized for success in this course and/or calculus.
Each of the Pythagorean identities relates two trig functions. You can express each one in terms of the other. For example, you could solve $\cos^2(\theta) + \sin^2(\theta) = 1$ for $\cos(\theta)$ as follows:

$$\cos^2(\theta) = 1 - \sin^2(\theta)$$

$$\cos(\theta) = \pm \sqrt{1 - \sin^2(\theta)}.$$  The choice of plus or minus depends on the quadrant: First recall that (like any other radical) $\sqrt{1 - \sin^2(\theta)}$ is positive. Therefore $\cos(\theta) = \sqrt{1 - \sin^2(\theta)}$ in Quadrants 1 and 4, where $\cos(\theta)$ is positive, while $\cos(\theta) = -\sqrt{1 - \sin^2(\theta)}$ in Quadrants 2 and 3, where $\cos(\theta)$ is negative.

**Example 4:** If $\theta$ is a Quadrant 3 angle, express $\tan(\theta)$ in terms of $\sin(\theta)$.

**Solution:** From above, we know that $\cos(\theta) = -\sqrt{1 - \sin^2(\theta)}$ in Quadrant 3. Therefore

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sin(\theta)}{-\sqrt{1 - \sin^2(\theta)}}$$

and so in Q3: $$\tan(\theta) = -\frac{\sin(\theta)}{\sqrt{1 - \sin^2(\theta)}}$$

Check your answer: In Quadrant 3, $\sin(\theta)$ is negative in Q3 while $\tan(\theta)$ is positive. Therefore both sides of the boxed answer have the same sign. If they would have different signs, you would need to go back and check for an error.
Evaluating trigonometric functions

Example 5: If $\sec(\theta) = -\frac{5}{3}$ and $\theta$ is in Quadrant 2, find the other 5 trig functions of $\theta$.

Solution: If the terminal line of $\theta$ goes from the origin to $P(x, y)$, located distance $r$ from the origin, then $\sec(\theta) = \frac{r}{x} = -\frac{5}{3}$. Choose any $r$ and $x$ that satisfy this equation, but remember that $r$ is positive. The easiest solution is $r = 5$ and $x = -3$. But $x^2 + y^2 = r^2$. Thus $(-3)^2 + y^2 = 5^2$; $9 + y^2 = 25$ and so $y = \pm 4$. But $(x, y)$ is in quadrant 2, where $y$ is positive, and so $y = 4$.

Therefore $x = -3, y = 4, r = 5$. Note that $\sec(\theta) = \frac{r}{x} = \frac{5}{-3}$ checks with the given information. The requested five trig functions are

\[
\begin{align*}
\cos(\theta) &= \frac{x}{r} = \frac{-3}{5} = -\frac{3}{5} \\
\sin(\theta) &= \frac{y}{r} = \frac{4}{5} \\
\tan(\theta) &= \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3} \\
\cot(\theta) &= \frac{x}{y} = \frac{-3}{4} = -\frac{3}{4} \\
csc(\theta) &= \frac{r}{y} = \frac{5}{4}
\end{align*}
\]
Example 6: If \( \theta \) is the radian measure of an angle, it is shown in Calculus III that if \(-\pi/2 \leq \theta \leq \pi/2\), then an approximate formula for cosine is

\[
\cos(\theta) \approx 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}
\]

You know that \( \cos(\pi/3) = \frac{1}{2} \).

Test the accuracy of the above formula with \( \theta = \pi/3 \approx 1.047 \).

Solution: Compute by hand with 3 decimal place accuracy, or (if you must) use a calculator to check that

\[
1 - \frac{1.047^2}{2} + \frac{1.047^4}{24} - \frac{1.047^6}{720} \approx 0.500
\]
The functions $y = \cos(\theta)$ and $y = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!}$ are graphed below. Note that the graphs look identical for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
How close to identical, really? The difference of the functions \( y = \cos(\theta) \) and 
\[ y = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \] 
is graphed below. Look at the scale to the left of the graph!

When \( \theta = \pi/3 \), the difference equals \( \approx 0.00003543467 \). In other words the formula for 
\( \cos(\pi/3) = 0.5 \) gives the correct answer with better than 99.99 per cent accuracy!
Quiz Review

Example 1: Find the cosines of the following angles: 150°, 225°, 300°

Example 2: Find the reference angle of 920°.

Example 3: Find the reference angle of $\frac{19\pi}{3}$.

Example 4: If $\theta$ is a Quadrant 3 angle, express $\tan(\theta)$ in terms of $\sin(\theta)$.

Example 5: If $\sec(\theta) = -\frac{5}{3}$ and $\theta$ is in Quadrant 2, find the other 5 trig functions of $\theta$. 