New feature: short written assignment

Please read the notes on Section 5.5, posted on Blackboard

At the start of class on Wednesday, November 20, please hand in neatly written solutions to the following textbook problems:
Section 5.5 (Page 411) 3a), 5c), and 7c)
Solving simple trigonometric equations

In Chapter 6.3, we started with an angle $\theta$ with $0 \leq \theta < 2\pi$ and figured out values of its trig functions. In this chapter, we work in reverse: given a trig function value, what is the angle whose trig function has that value? For some easy problems of this type, we can use the table to the right or the three triangles below. Memorize the triangles!

**Example 1:** Find an angle $\theta$ with $0 < \theta < 90^\circ$ and $\sin(\theta) = \frac{1}{2}$.

**Solution:** Consult the table at the right or the triangle pictures below to conclude $\theta = 30^\circ$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\sin(\theta)$</th>
<th>$\cos(\theta)$</th>
<th>$\tan(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\pi}{6}$</td>
<td>$30^\circ$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$45^\circ$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$</td>
<td>$60^\circ$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
Reminder: Angle $\theta$ is called acute whenever $0 < \theta < 90^\circ$.

**Reference angle principle (RAP):** For any angle $\theta$, the reference angle $\bar{\theta}$ is

- the unique acute angle with $\cos(\bar{\theta}) = |\cos(\theta)|$; and is also
- the unique acute angle with $\sin(\bar{\theta}) = |\sin(\theta)|$; and is also
- the unique acute angle with $\tan(\bar{\theta}) = |\tan(\theta)|$.

**Example 2:** Suppose $\sin(\theta) = -\frac{1}{2}$.  

a) Find $\bar{\theta}$.  

b) If $\theta$ is in Q3, find $\theta$.

**Solution:**

a) Since $|\sin(\theta)| = |-\frac{1}{2}| = \frac{1}{2} = \sin(30^\circ)$, RAP says that $\theta$ has reference angle $\bar{\theta} = 30^\circ$.

b) From the picture at the right, a quadrant 3 angle with reference angle $30^\circ$ is $180^\circ + 30^\circ = 210^\circ$.

However, any coterminal angle (obtained by adding or subtracting a multiple of a complete circle $= 360^\circ$) will also be in Q3 and have the same sine. Therefore the answer to the question is $\theta = 210^\circ \pm n \cdot 360^\circ; n = 0, 1, 2, ...$
Example 3: Suppose $\cos(\theta) = -\frac{\sqrt{3}}{2}$, where $0 < \theta < 2\pi$. Find $\theta$.

Solution:
- Since $|\cos(\theta)| = |-\frac{\sqrt{3}}{2}| = \frac{\sqrt{3}}{2} = \cos(30^\circ)$, RAP says that $\theta$ has reference angle $\bar{\theta} = 30^\circ$.
- Since $\cos(\theta) = -\frac{\sqrt{3}}{2}$ is negative, we see from ASTC that $\theta$ is in Q2 or Q3.
- The diagram at the right shows that a Q2 angle and its reference angle add to $180^\circ$. Therefore $\theta + 30^\circ = 180^\circ$ and so $\theta = 180^\circ - 30^\circ = 150^\circ$ is the solution in Q2.
Example 3: Suppose $\cos(\theta) = -\frac{\sqrt{3}}{2}$, where $0 < \theta < 2\pi$. Find $\theta$.

Solution:
• Since $|\cos(\theta)| = \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} = \cos(30^\circ)$, RAP says that $\theta$ has reference angle $\theta = 30^\circ$.
• Since $\cos(\theta) = -\frac{\sqrt{3}}{2}$ is negative, we see from ASTC that $\theta$ is in Q2 or Q3.
• The diagram at the right shows that a Q2 angle and its reference angle add to $180^\circ$. Therefore $\theta + 30^\circ = 180^\circ$ and so $\theta = 180^\circ - 30^\circ = 150^\circ$ is the solution in Q2.
• A similar diagram shows that a Q3 angle equals its reference angle $+ 180^\circ$, and so the Q3 solution is $180^\circ + 30^\circ = 210^\circ$.
• Summary: angles $150^\circ$ and $210^\circ$ are solutions. They should be converted to radians because the question asks for angles $\theta$ with $0 \leq \theta < 2\pi$.

Answer: $\theta = \frac{5\pi}{3}$ and $\theta = \frac{7\pi}{3}$.
In Section 6.3, we found the sine, cosine, or tangent of a given angle \( \theta \).
In the previous three slides, we did the reverse. For example, we were given \( \sin(\theta) = \frac{1}{2} \) and were asked to figure out the angle \( \theta \). We did this sort of thing in Chapter 2.7, where we studied the function \( f(x) = x^2 \).

The main points to remember from that discussion are:

- For any \( y > 0 \), solving the equation \( f(x) = y \) for \( x \) produces two solutions \( x = \pm \sqrt{y} \).
- If we restrict the domain of \( f \) to \( x \geq 0 \), then for each \( y \geq 0 \), the equation \( f(x) = y \) has a single solution \( \sqrt{y} \).
- The function \( f(x) = x^2 \) with restricted domain \( x \geq 0 \) has inverse function \( f^{-1}(x) = \sqrt{x} \), with domain \( x \geq 0 \).

The fact that the domains of \( f^{-1} \) and \( f \) in this example are the same is a coincidence. In general, \( f \) and \( f^{-1} \) will not have the same domain. The rule is: the domain of \( f^{-1} \) is the range of \( f \), while the range of \( f^{-1} \) is the domain of \( f \).
The goal of this chapter is to define and apply inverse functions for sine, cosine, and tangent. To do that, we need to restrict the domains of those functions. The inverse functions could be named $\tan^{-1}, \cos^{-1},$ and $\sin^{-1},$ but that notation can be confusing, since $\sin^{-1}(x)$ does NOT mean $\frac{1}{\sin(x)}$. Instead we use the names $\arccos, \arcsin, \arctan$ for these inverse functions, defined as follows:

**The inverse sine function**

*Given $x$ in $[-1, 1]$, then $\arcsin(x)$ is the unique angle $\theta$ with $\sin(\theta) = x$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.***

**The inverse cosine function**

*Given $x$ in $[-1, 1]$, then $\arccos(x)$ is the unique angle $\theta$ with $\sin(\theta) = x$ and $0 \leq \theta \leq \pi$.***

**The inverse tangent function**

*Given $x$ in $(-\infty, \infty)$, then $\arctan(x)$ is the unique angle $\theta$ with $\tan(\theta) = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.***

*The values of these inverse functions must be written as radians, not degrees.*
Example 4: Find a) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$  b) $\arctan(\sqrt{3})$  c) $\arccos\left(\frac{1}{\sqrt{2}}\right)$ d) $\arctan(1)$.

Solution: All of these problems are easy because the requested trig function value is positive. Since the values of $\arctan$, $\arccos$, and $\arcsin$ can always be an acute angle (between 0 and $\pi/2$), each answer is the unique acute angle with the given sine, cosine, or tangent. The table at the right, or the acute triangle pictures on the first slide, can be used to find these answers.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\sin(\theta)$</th>
<th>$\cos(\theta)$</th>
<th>$\tan(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\pi}{6}$ = 30°</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$ = 45°</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>1</td>
</tr>
<tr>
<td>$\frac{\pi}{3}$ = 60°</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\sqrt{3}$</td>
</tr>
</tbody>
</table>

Answers:
a) $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

b) $\arctan(\sqrt{3}) = \frac{\pi}{3}$ since $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.

c) $\arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ since $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$.

d) $\arctan(1) = \frac{\pi}{4}$ since $\tan\left(\frac{\pi}{4}\right) = 1$.

The problems on the following slides will be harder because the requested trig function value is in each case negative.
Example 5: Find $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$.

Solution: Begin by giving the answer a name: let $\theta = \arcsin\left(-\frac{\sqrt{3}}{2}\right)$.

- The definition of $\arcsin$ says that $\sin(\theta) = -\frac{\sqrt{3}}{2}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.
- ASTC: since $\sin(\theta)$ is negative, $\theta$ is an angle (i.e., has terminal line) in Q3 or Q4.
- Since we need $-90^\circ \leq \theta \leq 90^\circ$, we see $\theta$ is in Q4 or Q1.
- Conclusion: We are looking for angle $\theta$ in Q4 with $\sin(\theta) = -\frac{\sqrt{3}}{2}$.
- Since $|\sin(\theta)| = | -\frac{\sqrt{3}}{2} | = -\frac{\sqrt{3}}{2} = \sin(60^\circ)$, RAP says that $\theta$ has reference angle $\bar{\theta} = 60^\circ$.
- From the picture at the right, a (negative) quadrant 4 angle with reference angle $60^\circ$ is $-60^\circ = -\frac{\pi}{3}$.
Example 6: Find \( \arctan(-1) \).

Solution: Begin by giving the answer a name: let \( \theta = \arctan(-1) \).

- The definition of \( \arctan \) says that \( \tan(\theta) = -1 \) and \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \).
- ASTC: since \( \tan(\theta) \) is negative, \( \theta \) is an angle (i.e., has terminal line) in Q2 or Q4.
- Since we need \( -90^\circ \leq \theta \leq 90^\circ \), we see \( \theta \) is in Q4 or Q1.
- Conclusion: We are looking for angle \( \theta \) in Q4 with \( \tan(\theta) = -1 \).
- Since \( |\tan(\theta)| = |-1| = 1 = \tan(45^\circ) \), RAP says that \( \theta \) has reference angle \( \bar{\theta} = 45^\circ \).
- From the picture at the right, a (negative) quadrant 4 angle with reference angle \( 45^\circ \) is \( -45^\circ = -\frac{\pi}{4} \).
Example 7: Find $\arccos\left(-\frac{\sqrt{3}}{2}\right)$.

Solution: Begin by giving the answer a name: let $\theta = \arccos\left(-\frac{\sqrt{3}}{2}\right)$.

- The definition of $\arccos$ says that $\cos(\theta) = -\frac{\sqrt{3}}{2}$ and $0 \leq \theta \leq \pi$.
- ASTC: since $\cos(\theta)$ is negative, $\theta$ is an angle (i.e., has terminal line) in Q2 or Q3.
- Since we need $0 \leq \theta \leq 180^\circ$, we see $\theta$ is in Q1 or Q2.
- Conclusion: We are looking for angle $\theta$ in Q2 with $\cos(\theta) = -\frac{\sqrt{3}}{2}$.

Since $|\cos(\theta)| = \left|-\frac{\sqrt{3}}{2}\right| = -\frac{\sqrt{3}}{2} = \cos(30^\circ)$, RAP says that $\theta$ has reference angle $\theta = 30^\circ$.

- From the picture at the right, a quadrant 2 angle with reference angle $30^\circ$ is $180^\circ - 30^\circ = 150^\circ = \frac{5\pi}{6}$.
Example 8: Find $\arccos(-1)$, $\arccos(0)$, and $\arccos(1)$.

Solution:
- The easiest way to do this is to graph $x = \cos(\theta)$; $0 \leq \theta \leq \pi$
- Clearly $\cos(0) = 1$, $\cos(\frac{\pi}{2}) = 0$, and $\cos(\pi) = -1$.

Now exchange inputs and outputs:
Answer: $\arccos(1) = 0$, $\arccos(0) = \frac{\pi}{2}$, and $\arccos(-1) = \pi$.

Example 9: Find $\arcsin(-1)$, $\arcsin(0)$, and $\arcsin(1)$

Solution:
- The easiest way to do this is to graph $x = \sin(\theta)$; $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- Clearly $\sin\left(\frac{\pi}{2}\right) = 1$, $\sin(0) = 1$, and $\sin\left(-\frac{\pi}{2}\right) = -1$.

Now exchange inputs and outputs:
Answer: $\arcsin(1) = \frac{\pi}{2}$, $\arcsin(0) = 0$, and $\arcsin(-1) = -\frac{\pi}{2}$.
Quiz review

Example 1: Find an angle $\theta$ with $0 < \theta < 90^\circ$ and $\sin(\theta) = \frac{1}{2}$.

Example 2: Suppose $\sin(\theta) = -\frac{1}{2}$. a) Find $\bar{\theta}$. b) If $\theta$ is in Q3, find $\theta$.

Example 3: Suppose $\cos(\theta) = -\frac{\sqrt{3}}{2}$, where $0 < \theta < 2\pi$. Find $\theta$.

Example 4: Find a) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$ b) $\arctan(\sqrt{3})$ c) $\arccos\left(\frac{1}{\sqrt{2}}\right)$ d) $\arctan(1)$.

Example 5: Find $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$.

Example 6: Find $\arctan(-1)$.

Example 7: Find $\arccos\left(-\frac{\sqrt{3}}{2}\right)$.

Example 8: Find $\arccos(-1), \arccos(0)$, and $\arccos(1)$.

Example 9: Find $\arcsin(-1), \arcsin(0)$, and $\arcsin(1)$.