Exponential growth and decay

Suppose you deposit 100 dollars in a bank account that receives 5 percent interest, compounded annually. This means: at the end of each year, the bank adds to your account 5 percent of its value at the beginning of that year.

After 1 year, the account value is $100 + (5\% \text{ of } 100) = 100 + .05(100) = 1.05(100)$.

**Percentage increase principle:** Adding 5 percent to a number multiplies it by 1.05.

**Example 1:** My bank account starts off at 100 dollars and gets 5 percent interest, compounded annually. How much is in my account after 3 years? After $t$ years?

**Solution:** Let $N(t)$ be my account in dollars after $t$ years.

<table>
<thead>
<tr>
<th>My account starts at</th>
<th>$N(0) = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>After 1 year it is:</td>
<td>$N(1) = N(0)(1.05) = 100(1.05)$</td>
</tr>
<tr>
<td>After 2 years it is:</td>
<td>$N(2) = N(1)(1.05) = 100(1.05)^2$</td>
</tr>
<tr>
<td>After 3 years it is:</td>
<td>$N(3) = N(2)(1.05) = 100(1.05)^3$</td>
</tr>
<tr>
<td>After $t$ years:</td>
<td>$N(t) = 100(1.05)^t$</td>
</tr>
</tbody>
</table>

**Answer:** My account’s dollar value is $N(t) = N(0)(1.05)^t = 100(1.05)^t$ after $t$ years. After 3 years, it is $N(3) = N(0)(1.05)^3 = 100(1.05)^3 = 115.7625$ dollars.
Comment: The amount after 3 years is NOT just $100 + 5 + 5 + 5 = 115$ dollars. That would be the result of giving 5 percent interest each year on the original amount. In fact, however, each year’s interest is also getting 5 percent interest. This results after 3 years in an extra $0.7625$ dollars = an extra $76\frac{1}{4}$ cents.

Suppose the bank gives me $r$ percent interest, compounded annually.
- My account value is multiplied by $K = 1 + \frac{r}{100}$ at the end of each year.
- After $t$ years, the original amount $N(0)$ in my bank account is multiplied by $(1 + \frac{r}{100})^t$, and my account value is $N(t) = N(0)(1 + \frac{r}{100})^t$.

In the above discussion, the bank gives interest only once a year, and the formulas for $N(t)$ apply only when $t = 0, 1, 2, 3, \ldots$. We are interested in getting interest continuously (rough idea: once every second), not just once a year.

**Definition of exponential growth and decay rates**

A function of time $y = N(t)$ grows (or decays) exponentially means: $N(t) = N(0)K^t$ for all times $t \geq 0$. Here $K$ is positive and $N(0)$ is the initial value. If $K > 1$, then $K$ is the exponential growth rate. If $0 < K < 1$, then $K$ is the exponential decay rate.
Bank accounts that receive interest continuously grow exponentially, as do some bacteria populations. Samples of a radioactive element decay exponentially.

We have modeled exponential growth by \( N(t) = N(0)K^t \) where \( K \) is the exponential growth rate. If we let \( K = e^r \), then \( r = \ln K \) is called the relative growth rate for \( N(t) \).

**The relative growth rate for exponential growth**

- The relative growth rate for \( N(t) = N(0)K^t \) is \( r = \ln K \). Then \( N(t) = N(0)e^{rt} \).
- A bank account gets interest rate \( r \), compounded continuously, provided its value after \( t \) years is \( N(0)e^{rt} \).

**Example 2:** A population of bacteria doubles after 10 hours. Assume it grows exponentially. Find its relative growth rate.

**Solution:** Let \( N(t) \) be the population at time \( t \). We are given \( N(10) = 2N(0) \).

The growth formula is: \( N(t) = N(0)e^{rt} \).

Time \( t = 10 \): \( N(10) = N(0)e^{r \cdot 10} = 2N(0) = \) double the original amount

Divide by \( N(0) \): \( e^{r \cdot 10} = 2 \)

Take ln of both sides: \( r \cdot 10 = \ln 2 \quad \Rightarrow \quad r = \frac{\ln 2}{10} \)

**Answer:** The relative growth rate is \( \frac{\ln 2}{10} \approx .069 \approx 6.9\% \).
Example 3: My bank account starts off with 100 dollars at 2:00 P.M. and grows exponentially. At 4:00 P.M., its value is 110 dollars. Find its value at 10:00 P.M. Find its value at any time $t$.

Solution: Let $N(t)$ be the number of dollars in my bank account $t$ hours after 2:00 P.M. The problem tells us:

- $N(0) = 100$, the starting amount at 2:00, when we set $t = 0$.
- $N(2) = 110$, the amount at 4:00 P.M., which is 2 hours later, when $t = 2$.

We are asked to find out $N(t)$ at 10:00 P.M., when $t = 8$.

The formula for exponential growth is: $N(t) = N(0)K^t$

We are given $N(0) = 100$ : $N(t) = 100K^t$

Plug in $t = 2$:

$N(2) = 100K^2$

But we are given $N(2) = 110$ and so:

$110 = 100K^2$

Solve for $K$:

$\frac{110}{100} = 1.1 = K^2 \Rightarrow K = \sqrt{1.1} = 1.1^{\frac{1}{2}}$

The formula for $N(t)$ is therefore:

$N(t) = 100\left((1.1)^{\frac{1}{2}}\right)^t = 100(1.1)^{\frac{t}{2}}$

So at 10:00, when $t = 8$

$N(t) = 100(1.1)_{\frac{8}{2}} = 100(1.1)^{4}$

Answer: At time $t$ hours after 2:00 PM, $N(t) = 100(1.1)^{\frac{t}{2}}$.

At 10:00 P.M., my bank account’s value is $100(1.1)^{4}$ dollars.
Finding the growth factor $K$ in $N(t) = N(0)K^t$

We can shorten the solution to the last problem by examining what happens when you know the population $N(t)$ at initial time $t = 0$ as well as at some later time $t = T$.

Set $t = T$ in $N(t) = N(0)K^t$: 

$$N(T) = N(0)K^T$$

Divide by $N(0)$:

$$\frac{N(T)}{N(0)} = K^T$$

Raise to the power $\frac{1}{T}$:

$$\left(\frac{N(T)}{N(0)}\right)^{\frac{1}{T}} = (K^T)^{\frac{1}{T}} = K$$

Raise to power $t$ to find $K^t$:

$$\left(\left(\frac{N(T)}{N(0)}\right)^{\frac{1}{T}}\right)^t = K^t = \left(\frac{N(T)}{N(0)}\right)^{\frac{t}{T}}$$

Then the growth formula is

$$N(t) = N(0)K^t = N(0)\left(\frac{N(T)}{N(0)}\right)^{\frac{t}{T}}$$

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**Exponential growth formula when you know $N(0)$ and $N(T)$**

$$K = \left(\frac{N(T)}{N(0)}\right)^{1/T}$$

$$N(t) = N(0)K^t = N(0)\left(\frac{N(T)}{N(0)}\right)^{t/T}$$
Applying the exponential growth formula

**Example 4:** My bacteria colony starts off with 100 million bacteria at 2:00 P.M. and grows exponentially. At 4:00 P.M., there are 110 million bacteria. How many bacteria are there at 10:00 P.M.? at any time \( t \)?

**Solution:** We are given \( N(0) = 100 \) and \( N(T) = 110 \) when \( T = 2 \). Then

\[
\frac{N(T)}{N(0)} = \frac{110}{100} = 1.1 . \text{ Now use } N(t) = N(0) \left( \frac{N(T)}{N(0)} \right)^{t/T}
\]

**Answer:** \( t \) hours after 2:00 P.M. there are \( N(t) = 100(1.1)^{t/2} \) million bacteria. At 10:00 P.M., \( t = 8 \) and there are \( 100(1.1)^4 \) million bacteria.

**Example 5:** Solve \( N = CK^t \) for \( t \).

**Solution:** Divide by \( C \): \[ \frac{N}{C} = K^t \]

Take log of both sides: \[ \log \frac{N}{C} = \log K^t \]

Bring down the exponent: \[ \log \frac{N}{C} = t \log K \]

Divide by \( \log K \): \[ t = \frac{\log \frac{N}{C}}{\log K} = \frac{\log N - \log C}{\log K} \]
Example 6: My bank account starts off at 2:00 P.M. with 100 dollars and grows exponentially. At 4:00 P.M., its value is 110 dollars. When will its value be 140 dollars?

Solution: Let $N(t)$ be the number of dollars in my bank account $t$ hours after 2:00 P.M. The problem tells us:

- $N(0) = 100$, the starting amount at 2:00 P.M., when we set $t = 0$.
- $N(2) = 110$, the amount at 4:00 P.M., which is 2 hours later, when $t = 2$.

We are asked to find at which time $t$ (hours after 2:00 P.M.) does $N(t) = 140$.

We will apply the Exponential Growth formula with $T = 2$. 
The formula for exponential growth is: 

\[ N(t) = N(0) \left( \frac{N(T)}{N(0)} \right)^{\frac{t}{T}} \]

In our problem, \( T = 2 \) and \( N(T) = 110 \)

Given: \( \frac{N(2)}{N(0)} = \frac{110}{100} = 1.1 \)

Now find out when \( N(t) = 140 \):

\[ 140 = 100(1.1)^{\frac{t}{2}} \]

Divide by 100:

\[ \frac{140}{100} = 1.4 = (1.1)^{\frac{t}{2}} \]

Take log of both sides:

\[ \log 1.4 = \log (1.1)^{\frac{t}{2}} \]

Bring down the power:

\[ \log 1.4 = \frac{t}{2} \log 1.1 = \frac{t \log 1.1}{2} \]

Solve for \( t \):

\[ t = \frac{2 \log 1.4}{\log 1.1} \]

**Answer:** Since we set \( t = 0 \) at 2:00 P.M., the account value is 140 dollars when the time is \( t = \frac{2 \log 1.4}{\log 1.1} \approx 7.56 \) hours after 2:00 P.M.

On the next slide we solve the same problem but use \( N(t) = N(0)e^{rt} \), where \( r \) is the relative growth rate.
Example 7: My bank account starts off at 2:00 P.M. with value 100 dollars and grows exponentially. At 4:00 P.M., its value is 110 dollars. What is its relative growth rate? When will its value be 140 dollars?

Solution: Let \( N(t) \) be the dollar value of my account \( t \) hours after 2:00 P.M.

A formula for exponential growth is: \( N(t) = N(0)e^{rt} = 100e^{rt} \)

At time \( t = 2 \), the formula says: \( N(2) = 100e^{2r} \)

But we are given \( N(2) = 110 \) and so: \( 110 = 100e^{2r} \)

Divide by 100

\[
\frac{110}{100} = 1.1 = e^{2r}
\]

Take ln of both sides to solve for \( r \)

\[
\ln 1.1 = 2r
\]

The relative growth rate is therefore:

\[
r = \frac{1}{2} \ln 1.1
\]

Now find out when \( N(t) = 140 \):

\[
140 = 100e^{rt}
\]

Solve for \( t \):

\[
\frac{140}{100} = 1.4 = e^{rt}
\]

Take ln of both sides: \( \ln 1.4 = rt = \frac{1}{2}(\ln 1.1) \cdot t \)

Solve for \( t \):

\[
t = \frac{2\ln 1.4}{\ln 1.1}
\]

Answer: Since we set \( t = 0 \) at 2:00 P.M.,

The account value is 140 dollars at time \( t = \frac{2\ln 1.4}{\ln 1.1} \approx 7.56 \) hours after 2:00 P.M.

The account’s relative growth rate is \( r = \frac{1}{2} \ln 1.1 \approx 0.0477 \approx 4.77\% \).

The interest rate on the account is 4.77\%, compounded continuously.
**Example 8:** My bacteria colony starts off at 2:00 P.M. with 100 million bacteria and grows exponentially. At 4:00 P.M., there are 110 million bacteria. What is its relative growth rate? When will there be 140 million bacteria?

**Solution:** This is similar to the previous example, but here we use \( K = \left( \frac{N(T)}{N(0)} \right)^{1/T} \). Let \( N(t) \) be how many million bacteria there are \( t \) hours after 2:00 P.M.

The formula for exponential growth is: \( N(t) = N(0)K^t \) where \( K = \left( \frac{N(T)}{N(0)} \right)^{1/T} \).

- \( N(0) = 100; \ N(T) = N(2) = 110. \)
- The relative growth rate is therefore: \( r = \ln K = \frac{1}{2} \ln 1.1 \)
- Millions of bacteria at time \( t \) is then: \( N(t) = N(0)K^t = 100(1.1)^{t/2} \)
- Now find out when \( N(t) = 140: \)
  - Divide by 100
  - \( \frac{140}{100} = 1.4 = (1.1)^{t/2} \)
  - Take \( \ln \) of both sides:
    - \( \ln 1.4 = \frac{t}{2}(\ln 1.1) \)
    - \( t = \frac{2\ln 1.4}{\ln 1.1} \)

**Answer:** There are 140 million bacteria at time \( t = \frac{2\ln 1.4}{\ln 1.1} \) hours after 2:00 P.M. The bacteria colony’s relative growth rate is \( r = \frac{1}{2} \ln 1.1 \approx .0477 = 4.77\% \).
The time required for a radioactive substance (uranium, kryptonite, kolnidreinite, etc.) to decay exponentially to half its original amount is called the half-life of the element.

**Example 9:** A sample of element stuffium with mass 100 grams will decay exponentially to 70 grams 11 days from now.

a) Find the formula for \( N(t) \) = the mass of stuffium \( t \) days from now.

b) What is the half-life of stuffium?

c) How long does it take for the original sample to lose 65% of its mass?

**Solution to a):** Let \( N(t) \) = the number of grams of stuffium at time \( t \) days from now. The exponential growth/decay formula is \( N(t) = N(0)K^t = 100K^t \). We are given \( N(0) = 100 \) and \( N(11) = 70 \).

First find \( K \). We know that \( N(t) = 100K^t \)

At time \( t = 11 \), \( N(11) = 70 \) so:

\[
70 = 100K^{11} \Rightarrow K^{11} = \frac{7}{10} = .7
\]

Find \( K \):

\[
K = \sqrt[11]{.7} = (.7)^{\frac{1}{11}}
\]

Now we know

\[
N(t) = N(0)K^t = 100(.7)^{\frac{t}{11}}
\]

Power to a power:

\[
N(t) = 100(.7)^{\frac{t}{11}} \text{ since } \frac{1}{11}t = \frac{t}{11}
\]

**Answer to a):** \( N(t) = 100(.7)^{\frac{t}{11}} \)
Now that we know \( N(t) = 100(.7)^\frac{t}{11} \), we can do the other parts of the problem.

**Solution to b):** To find the half-life of stuffium, we want to know at what time \( N(t) \) is half of the starting amount 100 grams. That is, we want to find \( t \) when \( N(t) = 50 \).

We need to solve for \( t \):

\[
50 = 100(.7)^\frac{t}{11}
\]

Divide by 100:

\[
\frac{1}{2} = (.7)^\frac{t}{11}
\]

Take logs:

\[
\log \left( \frac{1}{2} \right) = \log \left( (.7)^\frac{t}{11} \right)
\]

Bring down the exponent

\[
-\log 2 = \frac{t}{11} \log .7 = \frac{t \log .7}{11}
\]

Solve for \( t \)

\[
t = -\frac{11 \log 2}{\log .7}
\]

**Answer to b):** The half-life of stuffium is \( -\frac{11 \log 2}{\log .7} \approx 21.4 \) years.

**Solution to c):** If the 100-gram sample loses 65% of its mass, the amount remaining will be \( 100 - .65(100) = .35(100) = 35 \) grams. You want to find the time \( t \) when \( N(t) = 100(.7)^\frac{t}{11} = 35 \). Replace 50 by 35 in the solution to b) given above.

**Answer to c):** Stuffium loses 65% of its mass after \( \frac{11 \log .35}{\log .7} \approx 32.4 \) years.
Quiz Review

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