We want to draw the graphs of polynomial functions $y = f(x)$.

- The *degree* of a polynomial in one variable $x$ is the highest power of $x$ that remains after terms have been collected.
- The degree of $y = x^2 + x$ is 2.
- The degree of $y = 2x^2 + 3x - x^2 - 3 - x^2$ is 1 because the terms collect to give $y = 3x - 3$.

The graph of a degree 1 polynomial (which can always be written as $y = mx + b$ with $m \neq 0$) is a slanted straight line with slope $m$. This was discussed in Chapter 1.

In this section we focus on *parabolas*, which are the graphs of quadratic (degree 2) polynomials $y = ax^2 + bx + c$, where $a \neq 0$.

- If $a > 0$, the parabola $y = ax^2 + bx + c$ opens upward (like a *cup*: $\bigcup$) and has a minimum point at $x = -\frac{b}{2a}$.
- If $a < 0$, the parabola $y = ax^2 + bx + c$ opens downward (like a *cap*: $\cap$) and has a maximum point at $x = -\frac{b}{2a}$.
You should already be familiar with the graph of \( y = x^2 \), which has a minimum point at the origin \((0, 0)\).
Similarly, the graph of $y = -x^2$ has a maximum point at the origin $(0, 0)$. 

\[
y = -x^2
\]
To analyze the graph of $y = x^2 + 2x$, write $y = x^2 + 2x = ax^2 + bx + c$. Thus $a = 1$ and $b = 2$. Since $a > 0$, the graph has a minimum point at $x = -\frac{b}{2a} = -\frac{2}{2} = -1$. At that point $y = x^2 + 2x = (-1)^2 + 2(-1) = 0$ and so the minimum point is $(x, y) = (-1, -1)$. 

Stanley Ocken

M19500 Precalculus Chapter 3.1: Quadratic polynomials
To analyze the graph of $y = 2x - x^2$, write $y = -x^2 + 2x = ax^2 + bx + c$. Thus $a = -1$ and $b = 2$. Since $a < 0$, the graph has a maximum point at $x = \frac{-b}{2a} = \frac{-2}{-2} = 1$. At that point $y = -x^2 + 2x = -(1)^2 + 2(1) = 1$ and so the maximum point is $(x, y) = (1, 1)$. 
Graphs of degree 3 polynomials (preview)

The graph of a degree 3 polynomial can have no maximum/minimum point, or one of each. Click to see the possibilities. We will study these examples in detail in Chapter 3.2.
Some degree 3 polynomials.

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Standard form equations for parabolas

In Chapter 1.8, we discussed how to convert an equation of a circle to a standard form that shows its center and radius.

Here we do the same thing for parabolas, which are the graphs of degree 2 polynomial equations $y = ax^2 + bx + c$ where $a \neq 0$. Every parabola has a single minimum or maximum point, called the vertex of the parabola.

Every parabola equation can be written in standard form $y = a(x - M)^2 + K$. Here $M$ is the x-coordinate of the maximum or minimum. If $a > 0$ the vertex of the parabola is the minimum point $(M, K)$. If $a < 0$ the vertex of the parabola is the maximum point $(M, K)$.

To understand why, just multiply out the standard form given above to get

$y = a(x^2 - 2Mx + M^2) + K$
$y = ax^2 - 2aMx + aM^2 + K$. Match this with
$y = ax^2 + bx + c$ to see that $b = -2aM$: the vertex is at $x = \frac{-b}{2a} = \frac{2aM}{2a} = M$. 
Example 1. Find the vertex of the standard form parabola \( y = -2(x + 3)^2 + 7 \). Is it a maximum point or a minimum point?

Solution: For any \( x \)-value, \( y \) is obtained by subtracting the positive number \( 2(x + 3)^2 \) from 7. Therefore \( y \) will be a maximum \((y = 7)\) when the subtracted quantity \( 2(x + 3)^2 \) equals 0, and that happens when \( x = -3 \).

Answer: The vertex is at \((-3, 7)\) and is a maximum point.

Example 2. Find the vertex of the standard form parabola \( y = 2(x + 3)^2 + 7 \). Is it a maximum point or a minimum point?

Solution: For any \( x \)-value, \( y \) is obtained by adding the positive number \( 2(x + 3)^2 \) to 7. Therefore \( y \) will be a minimum \((y = 7)\) when the added quantity \( 2(x + 3)^2 \) equals 0, and that happens when \( x = -3 \).

Answer: The vertex is at \((-3, 7)\) and is a minimum point.

Next we use completing the square to rewrite the general parabola equation \( y = ax^2 + bx + c \) in standard form. To do this, write \( ax^2 + bx = a(x^2 + \frac{b}{a}x) \) and then follow the procedure in Section 1.8, as follows.
Example 3: Rewrite $x^2 + 6x$ by completing the square

Plan: Add and subtract $(\frac{6}{2})^2 = 3^2 = 9$

$x^2 + 6x = x^2 + 6x + 9 - 9 = (x + 3)^2 - 9$

Solution: $x^2 + 6x = (x + 3)^2 - 9$

Example 4: Rewrite $x^2 - 7x$ by completing the square

Plan: Add and subtract $(\frac{-7}{2})^2 = \frac{49}{4}$

$x^2 - 7x = x^2 - 7x + \frac{49}{4} - \frac{49}{4}$

$= (x + \frac{-7}{2})^2 - \frac{49}{4}$

Solution: $x^2 - 7x = (x - \frac{7}{2})^2 - \frac{49}{4}$
Example 5: Rewrite the parabola equation \( x^2 + 6x + 7 \) in standard form. Find the vertex of the parabola. Is it a minimum or maximum point?

Method Add and subtract \( \left( \frac{6}{2} \right)^2 = 3^2 \).

Solution
\[
y = x^2 + 6x + 9 - 9 + 7
\]
\[
y = (x + 3)^2 - 9 + 7 = (x + 3)^2 - 2
\]

Find the vertex: Set \( x + 3 = 0 \) to get \( x = -3 \) and then \( y = -2 \).

Answer: The standard form equation is \( y = (x + 3)^2 - 2 \). The vertex is \((-3, -2)\). It is a minimum point since \( y = +x^2 + 6x + 7 \) has an \( x^2 \) term with positive coefficient.
Example 6: Rewrite \( y = -2x^2 + 6x - 4 \) in standard form.
Then find the vertex. Is it a maximum point or a minimum point?

Solution:

Factor out -2

\[ y = -2(x^2 - 3x) - 4. \quad \text{Equation (1)} \]

Complete the square to get \( x^2 - 3x = (x - \frac{3}{2})^2 - \frac{9}{4} \)

Rewrite Equation (1)

\[ y = -2((x - \frac{3}{2})^2 - \frac{9}{4}) - 4 \]

Multiply out

\[ y = -2(x - \frac{3}{2})^2 + 2\cdot\frac{9}{4} - 4 = -2(x - \frac{3}{2})^2 + \frac{9}{2} - 4 \]

Answer:

The standard form equation is \( y = -2(x - \frac{3}{2})^2 + \frac{1}{2} \)

The vertex is \( (x, y) = (\frac{3}{2}, \frac{1}{2}) \). It is a maximum point since the \( x^2 \) term in \( y = -2x^2 + 6x - 4 \) has a negative coefficient.
Example 7: Use the result of Example 6 to sketch the graph of the parabola.

\[ y = -2x^2 + 6x - 4 \]. Plot all intercepts and the vertex, and label the vertex as a maximum or a minimum.

Solution: From Example 6, we know that the vertex is at \((\frac{3}{2}, \frac{1}{2})\).

To find the y-intercept, set \(x = 0\) in \(y = -2x^2 + 6x - 4\) to get \(y = -4\). The graph meets the y-axis at \((0, -4)\).

To find the x-intercept, set \(y = 0\) in \(y = -2x^2 + 6x - 4\) and solve for \(x\) as follows:

\[
0 = -2x^2 + 6x - 4 = -2(x^2 - 3x + 2) = -2(x - 1)(x - 2) \quad \text{and so } x = 1 \text{ or } x = 2.
\]

The graph meets the x-axis at \((1, 0)\) and \((2, 0)\).

You need to choose reasonable scales to show the vertex \((\frac{3}{2}, \frac{1}{2})\) and the intercepts \((1, 0), (2, 0),\) and \((0, -4)\). The x-values of these points (in order) are 0, 1, \(\frac{3}{2}\), and 2. Therefore show x-values \(-1 \leq x \leq 4\).

The y-values of these points in order are \(-4, 0, 0, \frac{1}{2}\). Therefore choose y-values \(-5 \leq y \leq 1\).

To get full credit on the Chapter 3.1 Quiz, you must show ALL the features on the next slide.
1. Draw the grid, the axes, and insert x- and y-scale numbers. Write the equation at the top of the graph.
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2. Plot the x-intercept points (1,0) and (2,0).
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3. Plot the y-intercept point (0,-4).
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2. Plot the x-intercept points (1,0) and (2,0).

3. Plot the y-intercept point (0,-4).

4. Plot and label the maximum point (vertex) at \( \left( \frac{3}{2}, \frac{1}{2} \right) = (1.5, 0.5) \).
1. Draw the grid, the axes, and insert x- and y-scale numbers. Write the equation at the top of the graph.

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5-8. Connect the four points with a smooth curve. Continue past \(x = 2\) by reflecting the curve already drawn across the vertical line through the vertex.
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5-8. Connect the four points with a smooth curve. Continue past \(x = 2\) by reflecting the curve already drawn across the vertical line through the vertex.

9. Extend the graph a bit to the left of \(x = 0\).
1. Draw the grid, the axes, and insert $x$- and $y$-scale numbers. Write the equation at the top of the graph.

2. Plot the $x$-intercept points (1,0) and (2,0).

3. Plot the $y$-intercept point (0,-4).

4. Plot and label the maximum point (vertex) at \((\frac{3}{2}, \frac{1}{2}) = (1.5, 0.5)\).

5-8. Connect the four points with a smooth curve. Continue past $x = 2$ by reflecting the curve already drawn across the vertical line through the vertex.

9. Extend the graph a bit to the left of $x = 0$.

10. The arrows at the ends of the curve are required: they show behavior at infinity.

11. *Check that your graph has no sharp points.*