Combining functions: algebraic methods

Functions can be added, subtracted, multiplied, divided, and raised to a power, just like numbers or algebra expressions. If \( f(x) = x^2 \) and \( g(x) = x + 2 \), clearly 
\[ f(x) + g(x) = x^2 + x + 2. \]
Another way to say the same thing: If \( f(x) = x^2 \) and \( g(x) = x + 2 \), let \( h = f + g \) be the sum of the functions \( f \) and \( g \). Then we define the sum function \( f + g \) and write 
\[ (f + g)(x) = f(x) + g(x) = x^2 + x + 2. \]

A problem arises if \( f \) and \( g \) have different domains. In that case, the domain of the sum function \( f + g \) consists of all points that are in the domains of both \( f \) and \( g \).

**Example 1:** Let \( f(x) = \sqrt{x} \) and \( g(x) = \frac{1}{x} \). Let \( h(x) = f(x) + g(x) \). Find the formula for \( h(x) \) and find the domain of \( h(x) \).

**Solution:** The domain of \( f \) is all \( x \) with \( x \geq 0 \) while the domain of \( g \) is \( x \neq 0 \). If we want to add \( f(x) \) and \( g(x) \), these must both be defined, and so we need \( x \geq 0 \) and \( x \neq 0 \). These are both satisfied precisely when \( x > 0 \). Thus the sum function is defined by 
\[ h(x) = (f + g)(x) = \sqrt{x} + \frac{1}{x} \text{ with domain } (0, \infty). \]

A similar discussion holds for multiplying, dividing, and subtracting functions. Note that the domain of the quotient function \( f/g \) consists of numbers \( x \) that are in the domains of both \( f \) and \( g \) and provided \( g(x) \neq 0 \).
Composing functions

However, there is a completely different method of combining functions that is not connected with algebra. This method is called composition, and is based on letting the output of one function be the input for the other.

Here’s a “real-life” example. At starting time $t = 0$, the side length of a square is 10 cm. The square’s side length grows at a rate of 3 cm per second. Then the formula for the side is easy to figure out:

$$s = 10 + 3t$$

In this formula,

- $s$ depends on $t$.
- $t$ is the independent variable and $s$ is the dependent variable.
- $t$ is the input and $10 + 3t$ is the output.
- $s$ is a function of $t$. 
Let $t$ be time, $s$ the side length of a square, and $A$ the square’s area. Then

$$A = s^2 \quad \text{and} \quad s = 10 + 3t$$

Now $A$ depends on $s$, which in turn depends on $t$. To show how, just substitute $10 + 3t$ for $s$ in $A = s^2$. Of course, make sure to use parentheses. Thus $A = (10 + 3t)^2$.

**Example 2:** The volume of a spherical balloon with radius $r$ is $\frac{4}{3}\pi r^3$. Suppose the balloon is being filled with air, and its radius at time $t$ is $3 + 4t$. Find the balloon’s volume at time $t$.

**Solution:** $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (3 + 4t)^3$.

We need to repeat the above discussion, this time using function notation. Although it’s not obvious why this more complicated notion is necessary, please accept that it is needed for more advanced courses.
If $f$ and $g$ are functions, they can be composed to form a new function given by the formula $h(x) = f(g(x))$. In other words, if you start with an input $x$, then use the output of $g$ as the input of $f$, then the output for $h$ will be the output from $f$. This is a clumsy explanation. It says nothing more than the formula $h(x) = f(g(x))$. We say that $h$ is $f$ composed with $g$.

Some people like to draw a picture: Input $x \Rightarrow g \Rightarrow g(x) \Rightarrow f \Rightarrow f(g(x)) = output$

**Example 3:** Let $f(x) = x + 2$ and let $g(x) = x^2$. Find a) $f(g(x))$ and b) $g(f(x))$.

**Solution:**

a) $f(g(x)) = f(x^2) = (x^2) + 2 = x^2 + 2$

b) $g(f(x)) = g(x + 2) = (x + 2)^2$.

**Composition is not commutative.** In plain English: $f$ composed with $g$, given by $h(x) = f(g(x))$, will usually be unequal to and have no relationship with $g$ composed with $f$, given by $g(f(x))$. 
There is a simple rule for working with algebra expressions involving functions:

**Parenthesis Rule 3**

*When you substitute any expression for \( f(x) \) in a formula, replace \( f(x) \) by the expression enclosed in parentheses.*

In other words, when you substitute an expression for \( f(x) \), use parentheses just as you would when you substitute an expression for \( x \).

In section 2.4, we followed this rule (without calling attention to it) when we calculated average rates of change for a function. Now we need to concentrate on getting the algebra correct.

**Example 4:** Given \( f(x) = 3x - x^2 \), find and simplify \( \frac{f(x + h) - f(x)}{h} \).

**Solution:** see next slide.
Combining and composing functions

Start with the function definition  \( f(x) = 3x - x^2 \)
Substitute \( x + h \) for \( x \):
\[
\frac{f(x + h) - f(x)}{h} = \frac{(3(x + h) - (x + h)^2) - (3x - x^2)}{h}
\]

Here is the problem:
\[
\frac{f(x + h) - f(x)}{h}
\]

Substitute for \( f(x) \) and \( f(x + h) \):
\[
\frac{3x + 3h - x^2 - 2hx - h^2 - 3x + x^2}{h} = \frac{3h - 2hx - h^2}{h}
\]

Expand \((x + h)^2\):
\[
= \frac{(3x + 3h) - (x^2 + 2hx + h^2) - (3x - x^2)}{h}
\]

Distribute two minus signs:
\[
= \frac{3x + 3h - x^2 - 2hx - h^2 - 3x + x^2}{h}
\]

Collect terms
\[
= \frac{3h - 2hx - h^2}{h}
\]

Factor and cancel
\[
= \frac{h(3 - 2x - h)}{h} = 3 - 2x - h
\]

Back in Section 2.4, we studied the function \( f(t) = 96t - 16t^2 \), the height above ground of a rock thrown upward from the ground at speed 96 feet per second. For various small values of \( h \), we calculated the average rate of change of \( f(t) \) from \( t = 2 \) to \( t = 2 + h \) as \( \frac{f(2+h)-f(2)}{h} \). Let’s do the same thing here, but for arbitrary \( h \).
Example 5: Given \( f(t) = 96t - 16t^2 \), find and simplify \( \frac{f(2+h) - f(2)}{h} \).

Solution:

Start with the function definition \( f(t) = 96t - 16t^2 \).

Substitute 2 for \( t \): \( f(2) = 96(2) - 16(2)^2 = 128 \)

Substitute \( 2 + h \) for \( t \): \( f(2 + h) = 96(t + h) - 16(2 + h)^2 \)

Average rate of change is:

\[
\frac{f(2 + h) - f(2)}{h} = \frac{96(2 + h) - 16(2 + h)^2 - 128}{h}.
\]

Expand \((x + h)^2\):

\[
96(2) + 96h - 16(4 + 4h + h^2) - 128
\]

Distribute \(-16\):

\[
192 + 96h - 64 - 64h - 16h^2 - 128
\]

Collect terms:

\[
\frac{32h - 16h^2}{h}
\]

Factor and cancel:

\[
\frac{h(32 - 16h)}{h} = 32 - 16h
\]

Conclusion: the average velocity of the rock from time \( t = 2 \) to time \( t = 2 + h \) is \( 32 - 16h \) feet/second. Thus the rock’s instantaneous velocity at \( t = 2 \), obtaining by letting \( h \) approach 0, is 32 feet/second.
Example 6: Given $f(x) = 3x - x^2$ and $g(x) = 4x + 7$, find $f(g(x))$ and $g(f(x))$ and rewrite each as a simplified product.

Solution:

- $f(g(x)) = f(4x + 7) = 3(4x + 7) - (4x + 7)^2 = (4x + 7)(3 - (4x + 7)) = (4x + 7)(3 - 4x - 7) = (4x + 7)(-4x - 4) = -4(x + 1)(4x + 7)$

- $g(f(x)) = 4(3x - x^2) + 7 = -4x^2 + 12x + 7$. To decide whether this factors, calculate the discriminant $D = b^2 - 4ac = 144 - 4(-4)(7) = 144 - 112 = 32$. Since $\sqrt{D} = \sqrt{32} = 4\sqrt{2}$ is not a whole number, the quadratic factoring criterion from Section 1.5 tells us that $g(f(x)) = -4x^2 + 12x + 7$ does not factor and is therefore a simplified product.

Note that $g(f(x))$ and $f(g(x))$ are unrelated: in general, composition is not commutative.
Example 7: Given \( f(x) = \frac{3x + 2}{5} \) and \( g(x) = \frac{5x - 2}{3} \), find \( f(g(x)) \) and \( g(f(x)) \) and rewrite each as a simplified sum.

Solution:

- \( f(g(x)) = f \left( \frac{5x - 2}{3} \right) = \frac{3 \left( \frac{5x - 2}{3} \right) + 2}{5} = \frac{5x - 2 + 2}{5} = \frac{5x}{5} = x \)

- \( g(f(x)) = g \left( \frac{3x + 2}{5} \right) = \frac{5 \left( \frac{3x + 2}{5} \right) - 2}{4} = \frac{3x + 2 - 2}{3} = \frac{3x}{3} = x \)

Note that \( g(f(x)) \) and \( f(g(x)) \) are both equal to \( x \). If you start with any number, apply either function, then apply the other function, you get back to the original number. We say that the functions \( f \) and \( g \) undo each other. The more technical language is that they are inverse functions, which we will study in detail in Chapter 2.7. The problem on the next slide is a warmup for the material in that chapter.
Example 8: Solve \( y = \frac{3x-2}{5x+7} \) for \( x \) and check your answer.

Solution:
The given equation is \( y = \frac{3x-2}{5x+7} \) 
Multiply both sides by \( 5x + 7 \) \( (5x + 7)y = 3x - 2 \)
Remove parentheses \( 5xy + 7y = 3x - 2 \)
Terms with \( x \) to left side \( 5xy - 3x + 7y = -2 \)
Terms with \( y \) on right side \( 5xy - 3x = -2 - 7y \)
Factor out \( x \) on left side \( x(5y - 3) = -2 - 7y \)
Divide by \( 5y - 3 \) \( x = \frac{-2 - 7y}{5y - 3} \)

Multiply answer by \( \frac{-1}{-1} \) \( x = \frac{7y + 2}{-5y + 3} \)

The check is on the following slide.
The given equation is

\[ y = \frac{3x-2}{5x+7} \]

We want to check the answer

\[ x = \frac{7y+2}{5y+3} \]

Plug in the answer

\[ y = \frac{3 \left( \frac{7y+2}{5y+3} \right) - 2}{5 \left( \frac{7y+2}{5y+3} \right) + 7} \]

Rewrite -2 and 7 as fractions

\[ y = \frac{3 \left( \frac{7y+2}{5y+3} \right) - 2 \left( \frac{-5y+3}{5y+3} \right)}{5 \left( \frac{7y+2}{5y+3} \right) + 7 \left( \frac{-5y+3}{5y+3} \right)} \]

Combine fractions

\[ y = \frac{21y+6+10y-6}{35y+10-35y+21} \]

Multiply by \( \frac{7y+2}{7y+2} \)

\[ y = \frac{31y}{31} \ldots YES! \]