THE CITY COLLEGE DEPARTMENT OF MATHEMATICS
MATHEMATICS 34600 FINAL EXAMINATION SPRING 2007

PART I. ANSWER ALL EIGHT QUESTIONS (80 points).

1. a) Write $\begin{bmatrix}1\\3\\7\end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}2\\1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

b) If $L : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that
$L\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \begin{bmatrix}3\\2\end{bmatrix}$, $L\left(\begin{bmatrix}2\\1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\4\end{bmatrix}$, and $L\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix}$, find $L\left(\begin{bmatrix}1\\3\\7\end{bmatrix}\right)$.

2. Find the inverse of $\begin{bmatrix}1 & 0 & 1\\ -2 & 1 & -2\\ 2 & -3 & 1\end{bmatrix}$, or show that it does not exist.

3. Let V be the vector space with basis $S = \{\sin t, \cos t\}$, and let $L : V \to V$ be the linear transformation defined by $L(f(t)) = f(t) + 2f'(t) + 3f''(t)$ for $f(t) \in V$. Find the matrix $[L]_S$ representing L with respect to S, and *use it* to find $L(3\sin t + \cos t)$.

4. If $A = \begin{bmatrix}1 & 3 & 2\\ 2 & 6 & 1\end{bmatrix}$, a) Find a basis for the nullspace of A.

b) Find a basis for the column space of A.

c) Find a basis for the row space of A *consisting of rows of* A.

5. Express $A = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 2\\ 0 & 0 & 3\end{bmatrix}$ as a product of elementary matrices. (Hint: first put A in row-echelon form by a sequence of elementary row operations.)

6. a) Evaluate the determinant $\begin{vmatrix}0 & 7 & 1 & 0 & -1\\ 3 & 4 & 6 & 0 & 5\\ 0 & 0 & 4 & 0 & 0\\ 0 & 5 & 8 & 0 & 0\\ 7 & 1 & -5 & 2 & 6\end{vmatrix}$.

b) In the expansion of the determinant $\begin{vmatrix}a & b & c & d & e\\ f & g & h & i & j\\ k & l & m & n & p\\ q & r & s & t & u\\ v & w & x & y & z\end{vmatrix}$, is the term $dgksz$ preceded by a + or a − sign? Explain.

7. *Use Cramer's Rule* to find x_3 if $x_1 + x_2 - x_3 = 2,\ x_1 + 4x_3 = 0,\ 2x_2 - 11x_3 = -1$. (No credit for any other method.)

8. a) Find the eigenvalues and corresponding eigenvectors for the linear operator on $\mathbb{R}^3$ whose matrix representation is $\begin{bmatrix}2 & 0 & 0\\ 1 & 2 & 0\\ 3 & -1 & 1\end{bmatrix}$.

b) Can this linear transformation be diagonalized? Explain.

PART II. ANSWER ONLY <u>TWO</u> OUT OF THE THREE QUESTIONS (20 points).

9. If $S = \{v_1, v_2\}$ is a basis of a vector space V, show that the set $\{2v_1 + v_2, v_1 - v_2\}$ is also a basis of V.

10. a) Define: λ is an eigenvalue of a linear operator $L : V \to V$.

b) If $L : V \to V$ is a linear operator and λ is an eigenvalue of L, prove that the set of all eigenvectors belonging to λ, together with the 0-vector, forms a subspace of V.

11. Let $L : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation.

a) Is it possible for L to be surjective? If so, give an example; if not, explain why.

b) Is it possible for L to be injective? If so, give an example; if not, explain why.