MATH 392 Final Exam, May 16, 2008

Answer all questions 1–7 and 3 of the 4 questions 8–11.

Question 1 (a) Find the inverse of the following 3 x 3 matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ -2 & 3 & -1 \end{pmatrix}$$

(b) Use the inverse matrix found in part (a) to solve the system of equations:

$$\begin{aligned} x + y + 4z &= 4 \\ y + z &= 8 \\ -2x + 3y - z &= -6 \end{aligned}$$

Question 2 (a) Calculate the eigenvalues and corresponding eigenvectors of

$$\mathbf{A} = \begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix}$$

(b) Give the general solution to the system of the ordinary equations:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Question 3 Evaluate the line integral

$$\int_C 6x^2\, ds,$$

where C is the part in the first quadrant of the circle of radius 4 in the xy plane centered at the origin.

Question 4 Let

$\vec{F} = (y^2 e^{xy} + 6xy^2 z)\vec{i} + (e^{xy} + xye^{xy} + 12z + 6x^2 yz)\vec{j} + (12y + 3x^2 y^2)\vec{k}$

(a) Show that $\vec{F}$ is conservative; that is, find a scalar function f such that $\vec{F} = \nabla f$.

(b) Find the work done by $\vec{F}$ along the path from $(0, 1, 0)$ to $(3, 4, -1)$ to $(1, -1, 3)$ along two straight segments.

Question 5 (a) Calculate the determinant of 4×4 matrix $\mathbf{A}$:

$$\begin{pmatrix} 0 & -2 & -2 & 4 \\ 1 & 4 & 6 & -3 \\ 1 & 6 & -4 & 5 \\ 2 & 12 & 1 & 6 \end{pmatrix}.$$

(b) Calculate the determinant of the matrix $\mathbf{B} = -3\mathbf{A}^3$.

Question 6 Find the general solution to the following linear system:

$$\begin{array}{rrrrrrl} 3u & -6v & +2w & +4x & -y & & = 2 \\ u & -2v & +w & +x & & & = 1 \\ u & -2v & & +2x & +y & +z & = 6 \\ u & -2v & & +2x & & & = 3 \end{array}$$

Express the general solution as a linear combination of column vectors.

Question 7 Calculate the surface integral to find the flux

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS,$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 9$ with $z \leq 0, y \geq 0$ and

$\vec{F} = x\vec{i} + y\vec{j} + (z+2)\vec{k}$, and $\vec{n}$ is the unit normal vector field along S directed away from the origin.

Question 8 Evaluate the surface integral

$$\iint_S 4yz\,dS$$

where S is part of an elliptic paraboloid parameterized as $\vec{r}(u,v) = < u^2, u\sin(v), u\cos(v) >$, with $0 \le u \le 2$ and $0 \le v \le \pi$. $x = y^2 + z^2$

Question 9 Find the work done by the vector field

$$\vec{F} = (e^x + x^2 y)\vec{\imath} + (e^y - xy^2)\vec{\jmath}$$

around the circle of radius 3 centered at the origin travelled clockwise.

Question 10 Let C be the intersection curve of the surfaces $z = 3x - 7$ and $x^2 + y^2 = 1$, oriented clockwise as seen from above. Let $\vec{F} = (4z - 1)\vec{\imath} + 2x\vec{\jmath} + (5y + 1)\vec{k}$. Compute the work integral $\int_c \vec{F} \cdot d\vec{r} = \int_c \vec{F} \cdot \vec{T}\,ds$ two ways:
(a) directly as a line integral
(b) as a double integral, using Stokes' Theorem.

Question 11 Let T be the part of the surface $z = 9 - x^2 - y^2$ which lies above the xy plane, with upward unit normal. Let B be the disk of radius 3 in the xy plane centered at the origin with downward unit normal. Find the total combined flux of the vector field $\vec{F} = (2x + 9y - 2yz)\vec{\imath} + (3x + y - 5z)\vec{\jmath} + (x^2 + y^2 + 2z^2)\vec{k}$ across T and B in the directions of their given normals.

Math 39200, Spring 2008 Final Exam Solutions

1) (a)
$$\left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -2 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{2R_1+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 5 & 7 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-5R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 & -5 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3\to R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -5/2 & 1/2 \end{array}\right]$$

$$\xrightarrow[-R_3+R_2\to R_2]{4R_3+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -3 & 10 & -2 \\ 0 & 1 & 0 & -1 & 7/2 & -1/2 \\ 0 & 0 & 1 & 1 & -5/2 & 1/2 \end{array}\right] \xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \overbrace{\begin{matrix} -2 & 13/2 & -3/2 \\ -1 & 7/2 & -1/2 \\ 1 & -5/2 & 1/2 \end{matrix}}^{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

(b) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 4 \\ 8 \\ -6 \end{pmatrix} = \begin{pmatrix} -2 & 13/2 & -3/2 \\ -1 & 7/2 & -1/2 \\ 1 & -5/2 & 1/2 \end{pmatrix}\begin{pmatrix} 4 \\ 8 \\ -6 \end{pmatrix} = \begin{pmatrix} 53 \\ 27 \\ -19 \end{pmatrix}$$

② (a) $$A = \begin{pmatrix} 6 & 3 \\ -1 & 2 \end{pmatrix} \Rightarrow \det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 3 \\ -1 & 2-\lambda \end{vmatrix}$$

$$= (6-\lambda)(2-\lambda) + 3 = \lambda^2 - 8\lambda + 15 = (\lambda-3)(\lambda-5)$$

eigenvalues of A: $\lambda = 3$ and $\lambda = 5$

$\underline{\lambda = 3}$:

$$A - \lambda I = \begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix} \xrightarrow[\downarrow]{\text{row reduces to}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{matrix} x_1 + x_2 = 0 \\ x_1 = -x_2 \end{matrix}$$

$$\Rightarrow \text{eigenvectors: } \begin{pmatrix} -r \\ r \end{pmatrix}, r \in \mathbb{R} \Rightarrow y_I = \begin{pmatrix} -re^{3t} \\ re^{3t} \end{pmatrix}$$

$\underline{\underline{\lambda = 5}}$: $$A - \lambda I = \begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix} \xrightarrow[\downarrow]{\text{row reduces to}} \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{matrix} x_1 + 3x_2 = 0 \\ x_1 = -3x_2 \end{matrix}$$

$$\Rightarrow \text{eigenvectors: } \begin{pmatrix} -3s \\ s \end{pmatrix}, s \in \mathbb{R} \Rightarrow y_{II} = \begin{pmatrix} -3se^{5t} \\ se^{5t} \end{pmatrix}$$

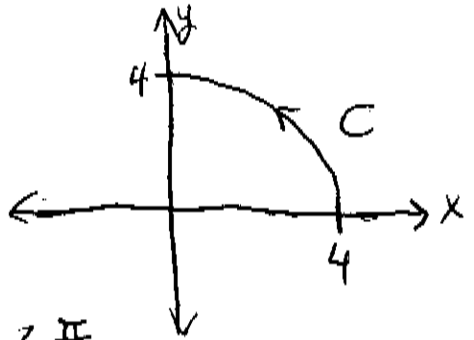
(b)

$$\Rightarrow V = y_I + y_{II} = \begin{pmatrix} -re^{3t} + 3se^{5t} \\ re^{3t} + se^{5t} \end{pmatrix}$$ gives the general solution to the system of ODE's

$x =$ "y_1"

$y =$ "y_2"

③ $\int_C 6x^2 ds$



$r(t)=\langle 4\cos t, 4\sin t\rangle,\ 0\le t\le \frac{\pi}{2}$

$\Rightarrow r'(t)=\langle -4\sin t, 4\cos t\rangle \Rightarrow \|r'(t)\| = \sqrt{16} = 4$

$\Rightarrow ds = 4dt \Rightarrow \int_C 6x^2 ds = \int_0^{\pi/2} 6(4\cos t)^2 \cdot 4\,dt = 384\int_0^{\pi/2}\cos^2 t\,dt$

$= 192\int_0^{\pi/2}(1+\cos(2t))dt = 192\left(\frac{\pi}{2}+\frac{1}{2}\sin(2t)\Big]_0^{\pi/2}\right) = \boxed{96\pi}$

④ (a) $\text{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} =$

where $P = y^2e^{xy}+6xy^2z$

$Q = e^{xy}+xye^{xy}+12z+6x^2yz$

$R = 12y+3x^2y^2$

$= \langle \underbrace{(12+6x^2y)-(12+6x^2y)}_{0}, -\underbrace{(6xy^2-6xy^2)}_{0},$

$\underbrace{ye^{xy}+y(x\cdot ye^{xy}+e^{xy}\cdot 1)+12xyz-(y^2\cdot xe^{xy}+e^{xy}\cdot 2y+12xyz)}_{0}\rangle$

$= \vec{0} \Rightarrow$ Since $\text{curl} F = \vec{0}$, F is conservative on $\mathbb{R}^3$.

So we are not wasting our time by searching for a potential function.

$$\frac{\partial f}{\partial x} = y^2e^{xy}+6xy^2z \Rightarrow$$

$$f(x,y,z) = ye^{xy}+3x^2y^2z+g(y,z)$$

$$\Rightarrow \frac{\partial f}{\partial y} = y\cdot xe^{xy}+e^{xy}\cdot 1+6x^2yz+\frac{\partial g}{\partial y}$$

$$\Rightarrow \frac{\partial g}{\partial y} = 12z \Rightarrow g(y,z) = 12yz+h(z)$$

$$\Rightarrow f(x,y,z) = ye^{xy}+3x^2y^2z+12yz+h(z)$$

$$\Rightarrow \frac{\partial f}{\partial z} = 3x^2y^2+12y+h'(z) \Rightarrow h'(z)=0 \Rightarrow h(z)=k,$$

a constant

$$\Rightarrow \boxed{f(x,y,z) = ye^{xy}+3x^2y^2z+12yz+k}$$ ← general potential function

(b) Since F is conservative we can evaluate its potential function at the endpoints of the given path and take the difference.

In symbols, $\int_C F\cdot dr = f(1,-1,3)-f(0,1,0)$ (let $k=0$ in answer to (a))

$$= (e^{-1}+9-36)-(1) = \boxed{\frac{1}{e}-28}$$

(You may convince yourself by evaluating the line integral directly.)

⑤(a) $\det(A) = \begin{vmatrix} 0 & -2 & 0 & 0 \\ 1 & 4 & 2 & 5 \\ 1 & 6 & -10 & 17 \\ 2 & 12 & -11 & 30 \end{vmatrix}$ where I have added $(-1)C_2$ to C_3 & $2C_2$ to C_4

We can evaluate this determinant by expanding along the first row to get

$$(-2)(-1)^{1+2}\begin{vmatrix} 1 & 2 & 5 \\ 1 & -10 & 17 \\ 2 & -11 & 30 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 5 \\ 0 & -12 & 12 \\ 0 & -15 & 20 \end{vmatrix} = 2(60) = \boxed{120}$$

(b) $\det(B) = \det(-3A^3) = (-3)^4(\det(A))^3 = \boxed{81(120)^3}$

⑥
$$\begin{array}{cccccc|c} u & v & w & x & y & z & \\ 3 & -6 & 2 & 4 & -1 & 1 & 2 \\ 1 & -2 & 1 & 1 & 0 & 0 & 1 \\ 1 & -2 & 0 & 2 & 1 & 1 & 6 \\ 1 & -2 & 0 & 2 & 0 & 0 & 3 \end{array}$$

$$\xrightarrow[\substack{-R_4+R_2\to R_2 \\ -R_4+R_3\to R_3}]{3R_4+R_1\to R_1}\begin{array}{cccccc|c} u & v & w & x & y & z & \\ 0 & 0 & 2 & -2 & -1 & 1 & -7 \\ 0 & 0 & 1 & -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & -2 & 0 & 2 & 0 & 0 & 3 \end{array}$$

$$\xrightarrow[\substack{\&\text{ then} \\ R_4\leftrightarrow R_3}]{R_1\leftrightarrow R_4}\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 & 0 & -2 \\ 0 & 0 & 2 & -2 & -1 & 1 & -7 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3\to R_3}\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 \end{array}\right]\xrightarrow[\substack{\&\text{ then} \\ (-1)R_3}]{R_3+R_4}\begin{array}{cccccc|c} u & & w & x & y & z & \\ 1 & -2 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{array}$$

general solution:

$u-2v=3 \Rightarrow u=2v+3$

$w=-2$

$x=$ anything

$y-z=3 \Rightarrow y=3$

$z=0$

$$\Rightarrow \begin{pmatrix} u \\ v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2v+3 \\ v \\ -2 \\ x \\ 3 \\ 0 \end{pmatrix}$$

$$= v\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \quad v, x \in \mathbb{R}$$

(linear combination of column vectors)

⑦ $F=\langle x, y, z+2\rangle$

Parametrization:

$r(\phi,\theta)=\langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi\rangle$

$\frac{\pi}{2}\leq\phi\leq\pi$ & $0\leq\theta\leq\pi$

$r_\phi=\langle 3\cos\phi\cos\theta, 3\cos\phi\sin\theta, -3\sin\phi\rangle$

$r_\theta=\langle -3\sin\phi\sin\theta, 3\sin\phi\cos\theta, 0\rangle$

$r_\phi \times r_\theta=\langle 9\sin^2\phi\cos\theta, 9\sin^2\phi\sin\theta, 9\sin\phi\cos\phi\rangle$ ← directed away from origin because $\sin\theta\geq 0$ for $0\leq\theta\leq 2\pi$ and so j-component points away from origin

$F(r(\phi,\theta))=\langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi+2\rangle$

$F(r(\phi,\theta))\cdot(r_\phi\times r_\theta)=27\sin^3\phi\cos^2\theta+27\sin^3\phi\sin^2\theta+27\cos^2\phi\sin\phi$
$+18\sin\phi\cos\phi$

$$= 27\sin^3\phi + 27\cos^2\phi\sin\phi + 18\sin\phi\cos\phi$$

$$\int_0^{\pi}\int_{\pi/2}^{\pi}\left(27\sin^3\phi + 27\cos^2\phi\sin\phi + 18\sin\phi\cos\phi\right)d\phi\, d\theta$$

$$= 9\int_0^{\pi}\int_{\pi/2}^{\pi}\left[3(1-\cos^2\phi)\sin\phi + 3\cos^2\phi\sin\phi + 2\sin\phi\cos\phi\right]d\phi\, d\theta$$

$$= 9\pi\left[3\int_{\pi/2}^{\pi}\sin\phi\, d\phi + \overbrace{3\int_{\pi/2}^{\pi}\cos^2\phi(-\sin\phi)\, d\phi - 3\int_{\pi/2}^{\pi}\cos^2\phi(-\sin\phi)\, d\phi}^{0} + 2\int_{\pi/2}^{\pi}\sin\phi\cos\phi\, d\phi\right]$$

$$= 9\pi\left[3\cos\phi\Big]_{\pi/2}^{\pi} + \sin^2\phi\Big]_{\pi/2}^{\pi}\right] = 9\pi(-3-1) = \boxed{-36\pi}$$

~~Compare this to the following:~~

~~$\operatorname{div}F = 1+$ [illegible] $\Rightarrow \iint_S F\cdot dS = \iiint_E \operatorname{div}F\, dV$~~

Ⓠ: Why can't you use the divergence theorem here?

⑧ $\iint_S 4yz\, dS \Rightarrow f(x,y,z) = 4yz$

$f(r(u,v)) = 4(u\sin v)(u\cos v) = 4u^2\sin v\cos v$

$r(u,v) = \langle u^2, u\sin v, u\cos v\rangle,\ 0 \le u \le 2,\ 0 \le v \le \pi$

$\Rightarrow r_u = \langle 2u, \sin v, \cos v\rangle$

& $r_v = \langle 0, u\cos v, -u\sin v\rangle \Rightarrow r_u \times r_v = \langle -u, 2u^2\sin v, 2u^2\cos v\rangle$

$$\|r_u \times r_v\| = \sqrt{(-u)^2 + (2u^2\sin v)^2 + (2u^2\cos v)^2}$$

$$= \sqrt{u^2(1+4u^2)} = u\sqrt{1+4u^2} \text{ because } \underline{\underline{u \geq 0}}$$

$$\iint_S 4yz\,dS = \int_0^{\pi}\int_0^{2} (4u^2\sin v\cos v)\,u\sqrt{1+4u^2}\,du\,dv$$

$$= 4\int_0^{\pi}\sin v\cos v\,dv \int_0^2 u^3\sqrt{1+4u^2}\,du \quad \left(\begin{array}{l} t = 1+4u^2 \Rightarrow u^2 = \frac{t-1}{4} \\ dt = 8u\,du \Rightarrow \frac{1}{8}dt = u\,du \end{array}\right)$$

$$= \boxed{0}$$

⑨ $F = \langle e^x + x^2y,\ e^y - xy^2 \rangle$

Direct Evaluation

$r(t) = \langle 3\cos t, 3\sin t\rangle,\ 0 \leq t \leq 2\pi$ ← This goes around C clockwise.

$$\int_C F\cdot dr = -\int_0^{2\pi} F(r(t))\cdot r'(t)\,dt$$

$$F(r(t)) = \langle e^{3\cos t} + 27\cos^2 t\sin t,\ e^{3\sin t} - 27\sin^2 t\cos t \rangle$$

& $r'(t) = \langle -3\sin t, 3\cos t\rangle$

$$F(r(t))\cdot r'(t) = -3\sin t\, e^{3\cos t} - 81\cos^2 t\sin^2 t + 3\cos t\, e^{3\sin t} - 81\sin^2 t\cos^2 t$$

$$\Rightarrow -\int_0^{2\pi} F(r(t))\cdot r'(t)\,dt = \int_0^{2\pi}\left(3\sin t\, e^{3\cos t} - 3\cos t\, e^{3\sin t} + 162\sin^2 t\cos^2 t\right)dt$$

$$= \left[-e^{3\cos t} - e^{3\sin t}\right]_0^{2\pi} + \frac{162}{4}\int_0^{2\pi}\overbrace{(1-\cos(2t))(1+\cos(2t))}^{1-\cos^2(2t)}\,dt$$

$$= -\left[(e^3+1)-(e^3+1)\right] + \frac{81}{2}\int_0 \frac{1-\cos(4t)}{2}\,dt$$

$$= \frac{81}{4}(2\pi) = \boxed{\frac{81\pi}{2}}$$

By Green's Theorem

$$P = e^x + x^2y \qquad\qquad \frac{\partial P}{\partial y} = x^2$$

$$\Longrightarrow$$

$$Q = e^y - xy^2 \qquad\qquad \frac{\partial Q}{\partial x} = -y^2$$

$$\int_C F\cdot dr = -\iint_D\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \iint (x^2+y^2)\,dA$$

$\uparrow$
C oriented clockwise

$$= \int_0^{2\pi}\int_0^3 r^2\cdot r\,dr\,d\theta$$

$$= 2\pi\cdot\left.\frac{r^4}{4}\right]_0^3 = \boxed{\frac{81\pi}{2}}$$

⑩ $z = 3x-7$ & $x^2+y^2=1$

$F = \langle 4z-1,\ 2x,\ 5y+1\rangle$

(a) $r(t) = \langle \cos t, \sin t, 3\cos t - 7\rangle,\ 0 \le t \le 2\pi$

$\Rightarrow F(r(t)) = \langle 12\cos t - 28 - 1,\ 2\cos t,\ 5\sin t + 1\rangle$

$r'(t) = \langle -\sin t, \cos t, -3\sin t\rangle$

$\Rightarrow F(r(t))\cdot r'(t) = -12\sin t\cos t + 29\sin t + 2\cos^2 t - 15\sin^2 t - 3\sin t$

$\Rightarrow \int_C F\cdot dr = \int_0^{2\pi}\left(12\sin t\cos t - 26\sin t - 2\cos^2 t + 15\sin^2 t\right)dt$

$= \int_0^{2\pi}\frac{13}{2}\,dt = \boxed{13\pi}$

(b) $\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4z-1 & 2x & 5y+1 \end{vmatrix} = \langle 5, 4, 2 \rangle$

I use the surface S which is the portion of the plane $z = 3x-7$ inside the cylinder $x^2+y^2=1$

A parametrization of this surface (without limits) is:

$r(x,y) = \langle x, y, 3x-7 \rangle$

$r_x = \langle 1, 0, 3 \rangle$
$r_y = \langle 0, 1, 0 \rangle$ $\Rightarrow r_x \times r_y = \langle -3, 0, 1 \rangle$

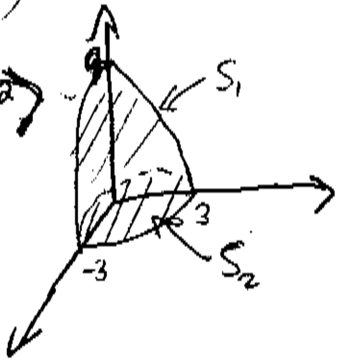
$\Rightarrow$ downward-pointing normal: $\langle 3, 0, -1 \rangle$
(want this because C is oriented clockwise)

$\Rightarrow \operatorname{curl} F(r(x,y)) \cdot (r_y \times r_x) = \langle 5,4,2 \rangle \cdot \langle 3,0,-1 \rangle = 15-2 = 13$

$\Rightarrow \iint_S \operatorname{curl} F \cdot dS = \iint 13\, dA = 13 \int_0^{2\pi}\int_0^1 r\, dr\, d\theta = \boxed{13\pi}$

($\int_C F \cdot dr$ is this by Stokes' Theorem)

⑪ $F = \langle 2x+9y-2yz, 3x+y-5z, x^2+y^2+2z^2 \rangle$



Direct evaluation

$\iint_S F \cdot dS = \iint_{S_1} F \cdot dS_1 + \iint_{S_2} F \cdot dS_2$

$\iint_{S_1} F \cdot dS_1$ Parametrization of S_1 (without limits)
$r(x,y) = \langle x, y, 9-x^2-y^2 \rangle$

$$r_x = \langle 1, 0, -2x \rangle$$
$$r_y = \langle 0, 1, -2y \rangle \Rightarrow r_x \times r_y = \langle 2x, 2y, 1 \rangle$$

points up because k-component > 0

$$\Rightarrow F(r(x,y)) = \langle 2x+9y-2y(9-x^2-y^2),\ 3x+y-5(9-x^2-y^2),$$
$$x^2+y^2+2(9-x^2-y^2)^2 \rangle$$

$$= \langle 2x-9y+2x^2y+2y^3,\ 3x+y+5x^2+5y^2-45,$$
$$x^2+y^2+2(9-x^2-y^2)(9-x^2-y^2) \rangle$$

$$= \langle 2x-9y+2x^2y+2y^3,\ 3x+y+5x^2+5y^2-45,$$
$$x^2+y^2+2(81-9x^2-9y^2-9x^2+x^4+x^2y^2$$
$$-9y^2+x^2y^2+y^4) \rangle$$

$$\Rightarrow F(r(x,y)) \cdot (r_x \times r_y) = 4x^2-18xy+4x^3y+4xy^3$$
$$+6xy+2y^2+10x^2y+10y^3-90y$$
$$+162+2x^4+4x^2y^2+2y^4-35x^2-35y^2$$

$$= 162-31x^2-33y^2-12xy-90y+10x^2y+10y^3+2x^4+2y^4+4x^2y^2$$
$$+4x^3y+4xy^3 = \circledast$$

Polan coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$$\iint \circledast \, dA = \int_0^{2\pi}\int_0^3 \Big[162-31r^2\cos^2\theta-33r^2\sin^2\theta+10r^3\sin^3\theta+$$
$$2r^4\cos^4\theta+2r^4\sin^4\theta+4r^4\cos^2\theta\sin^2\theta\Big]\, r\,dr\,d\theta$$

(Q: Why did I drop the other terms?)

$$= \int_0^{2\pi}\int_0^3 \left[162r - 31r^3\cos^2\theta - 33r^3\sin^2\theta + 10r^4\sin^3\theta + r(\sqrt{2}r^2\sin^2\theta + \sqrt{2}r^2\cos^2\theta)^2\right] dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 162r + 2r^5 - 31r^3\left(\frac{1+\cos(2\theta)}{2}\right) - 33r^3\left(\frac{1-\cos(2\theta)}{2}\right) + 10r^4(1-\cos^2\theta)\sin\theta\, dr\,d\theta$$

$$= 2\pi\cdot 162\cdot\frac{9}{2} + \frac{2\pi}{3}(3)^6 - \frac{31}{8}\cdot 2\pi\cdot(3)^4 - \frac{33}{8}\cdot 2\pi\cdot(3)^4$$

$$= 1458\pi + 2\pi(243) - \cancel{\frac{1053\pi}{4}}\ 8\cdot 2\pi(3)^4 = \boxed{648\pi}$$

~~= 1458π + 486π − 567π − 1944π − 567π [illegible]~~

$\boxed{\iint_{S_2} F\cdot dS_2}$ * Parametrization of S_2 (without limits)

$r(x,y) = \langle x, y, 0\rangle$

$n = \langle 0,0,-1\rangle$ ← downward-pointing

$F(r(x,y)) = \langle 2x+9y, 3x+y, x^2+y^2\rangle$

$F(r(x,y))\cdot n = -x^2-y^2$

$$\Rightarrow \iint_{S_2} F\cdot dS_2 = -\int_0^{2\pi}\int_0^3 r^2\cdot r\,dr\,d\theta = -\frac{2\pi}{4}r^4\Big]_0^3$$

$$= \boxed{-\frac{81\pi}{2}}$$

$$\Rightarrow \iint_S F\cdot dS = 648\pi - \frac{81\pi}{2} = \frac{1296\pi - 81\pi}{2} = \boxed{\frac{1215\pi}{2}}$$

Using Divergence Theorem

$\operatorname{div} F = \cancel{\ldots}\ 2+1+4z = 4z+3$

$$\Rightarrow \iint_S F\cdot dS = \int_0^{2\pi}\int_0^3\int_0^{9-r^2}(4z+3)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3} r\left[2z^2+3z\right]_0^{9-r^2} dr\,d\theta = 2\pi\int_0^3\left[r(2)(9-r^2)^2 + r(3)(9-r^2)\right]dr$$

$$= 2\pi\int_0^3\left[2r(81-18r^2+r^4)+27r-3r^3\right]dr$$

$$= 2\pi\int_0^3\left[189r-39r^3+2r^5\right]dr$$

$$= 2\pi\left(\frac{189r^2}{2}-\frac{39}{4}r^4+\frac{r^6}{3}\right)_0^3$$

$$= 2\pi\left(\frac{189(3)^2(6)-39(3)^4(3)+(3)^6\cdot 4}{12}\right)$$

$$= \frac{\pi}{6}(3)^5\left(\underbrace{42-39+12}_{=15}\right) = \boxed{\frac{1215\pi}{2}}$$

That was Certainly easier on the algebra!