Department of Mathematics **The City College of New York**

Math 39200 **Final Examination** **Spring 2006**

SHOW ALL WORK. Answer 5 questions from Part I and 2 from Part II

PART I. Answer 5 complete questions from this part. (14 points each)

1. a) Let A be the matrix $\begin{pmatrix} 0 & 1 & -2 & -2 \\ 0 & 1 & -2 & 1 \\ 1 & 2 & -1 & 0 \\ 3 & 2 & 1 & 1 \end{pmatrix}$.

Find the determinants of the following matrices: **i)** A , **ii)** A^{-1} and **iii)** $2A^3$

b) Let $B = \begin{pmatrix} 1 & 2 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$ Find **i)** BC and **ii)** $B^T C^T$.

2. Use Gaussian elimination to solve each of the following systems of equations. In part a), write the solution as a linear combination of vectors.

a) $\begin{cases} 3x - y + z = 0 \\ 2x + 4y - 2w = 2 \\ -7y + z + 3w = -3 \end{cases}$ **b)** $\begin{cases} x - y + 3z = 0 \\ 2x - y + 2z = 0 \\ x - z = 1 \end{cases}$

3. a) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 8 \\ 1 & 2 & 4 \end{pmatrix}$.

b) Use the matrix A^{-1} that you found in (a) to solve the system $\begin{cases} x + 2y + 3z = 1 \\ 4y + 8z = -1 \\ x + 2y + 4z = 4 \end{cases}$

4. a) Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$.

b) Let S be the surface parametrized by $x = u + 3$; $y = v + 2$; $z = uv^2$. Show that the point P with (x,y,z) co-ordinates (4,4,4) lies on S and find an equation of the tangent plane to S at point P.

5. a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 6 & 1 \\ -3 & 2 \end{pmatrix}$.

b) Use your answer to part a) to solve the following simultaneous differential equations for $y_1(t)$ and $y_2(t)$ subject to initial conditions $y_1(0) = 1$ and $y_2(0) = 2$:

$$\begin{cases} y_1' = 6y_1 + y_2 \\ y_2' = -3y_1 + 2y_2 \end{cases}$$

Please turn the page for the continuation of Part I and for Part II

Part I, continued

6. Let $\vec{F}(x,y,z) = < \frac{\ln y}{2\sqrt{x}} + yz, \frac{\sqrt{x}}{y} + xz, xy >$.

a) Find a potential function $f(x,y,z)$ for $\vec{F}$ so that $\nabla f = \vec{F}$ and

b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the straight line segment from $P(1,e,1)$ to $Q(4,e^2,2)$.

7. a) Let S be the surface described by $x^2 + y^2 = 1; 0 \le z \le 3$. Evaluate the integral $\iint_S z^2 dS$.

b) Find the length of the part of the parametrized curve $\vec{r}(t) = < \frac{t^2}{2}, \frac{2\sqrt{2}}{3} t^{3/2}, t >$ between the points P(0,0,0) and $Q(\frac{1}{2}, \frac{2\sqrt{2}}{3}, 1)$.

End of Part I. Make sure you answered five complete questions from this part.

PART II: Answer 2 complete questions from this part (15 points each).

8. Let R be the region in the x,y-plane bounded by the curves $x = y^2$ and $y = x - 2$. Find $\int_C -ydx + xdy$ (where C is the boundary of R, oriented clockwise)

a) directly, as a line integral **AND**
b) as a double integral, by using Green's Theorem.

9. Let S be the surface described by $z = x^2 + y^2$; $z \le 4$; $y \ge 0$ and let C be the boundary curve of S with the orientation of your choice. Let $\vec{F}(x,y,z) = < y,z,x >$. Find $\int_C \vec{F} \cdot d\vec{r}$

a) directly as a line integral **AND** **b)** as a double integral, by using Stokes' Theorem.

10. Let T be the solid bounded below by $z = 0$, bounded above by $y + z = 1$, and bounded on the side by $x^2 + y^2 = 1$. Let S be the boundary surface of T. Let $\vec{F}(x,y,z) = < x,y,z >$. Use the outward pointing normal vector to evaluate $\iint_S \vec{F} \cdot d\vec{S}$

a) directly as a surface integral **AND**
b) as a triple integral, by using the Divergence Theorem.

END OF EXAM. Please check that you answered five complete questions from Part I and two complete questions from Part II.

Math 39200, Spring 2006 Final Exam Solutions

Part I

① (a)

$$A=\begin{bmatrix}0 & 1 & -2 & -2\\ 0 & 1 & -2 & 1\\ 1 & 2 & -1 & 0\\ 3 & 2 & 1 & 1\end{bmatrix}$$

$$\xrightarrow{-3R_3+R_4\to R_4}\begin{bmatrix}0 & 1 & -2 & -2\\ 0 & 1 & -2 & 1\\ 1 & 2 & -1 & 0\\ 0 & -4 & 4 & 1\end{bmatrix}$$

expanding along first column

$$\Longrightarrow \det(A)=1(-1)^{3+1}\begin{vmatrix}1 & -2 & -2\\ 1 & -2 & 1\\ -4 & 4 & 1\end{vmatrix}\overset{(-1)R_1+R_2\to R_2}{=}\begin{vmatrix}1 & -2 & -2\\ 0 & 0 & 3\\ -4 & 4 & 1\end{vmatrix}$$

expanding along second row

$$= 3(-1)^{2+3}\begin{vmatrix}1 & -2\\ -4 & 4\end{vmatrix} = -3(-4)=12$$

i) $\boxed{12}$

ii) $\det(A^{-1})=\dfrac{1}{\det(A)}=\boxed{\dfrac{1}{12}}$

iii) $\det(2A^3)=(2)^4(\det(A))^3=\boxed{16(12)^3}$

(b) i) $BC=\begin{pmatrix}1 & 2 & -1\end{pmatrix}\begin{pmatrix}-2\\ 3\\ 1\end{pmatrix}=-2+6-1=\boxed{3}$

ii) $B^TC^T=\underset{3\times 1}{\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}}\underset{1\times 3}{\begin{pmatrix}-2 & 3 & 1\end{pmatrix}}=\boxed{\begin{pmatrix}-2 & 3 & 1\\ -4 & 6 & 2\\ 2 & -3 & -1\end{pmatrix}}$ ← 3×3

② (a)

$$\begin{array}{c} \begin{matrix} x & y & z & w & \end{matrix} \\ \left[\begin{array}{cccc|c} 3 & -1 & 1 & 0 & 0 \\ 2 & 4 & 0 & -2 & 2 \\ 0 & -7 & 1 & 3 & -3 \end{array}\right] \end{array} \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 0 & -2 & 2 \\ 3 & -1 & 1 & 0 & 0 \\ 0 & -7 & 1 & 3 & -3 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 3 & -1 & 1 & 0 & 0 \\ 0 & -7 & 1 & 3 & -3 \end{array}\right]$$

$$\xrightarrow{-3R_1 + R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & -7 & 1 & 3 & -3 \\ 0 & -7 & 1 & 3 & -3 \end{array}\right] \xrightarrow{-R_2 + R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & -7 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(NOTE: Technically did not complete Gaussian elimination as matrix is not in r.r.e.f.)

$$\Longrightarrow \begin{aligned} x + 2y - w &= 1 \\ -7y + z + 3w &= -3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x &= -2y + w + 1 \\ z &= 7y - 3w - 3 \end{aligned}$$

(I am treating y & w as free variables, x & z as leading.)

general solution

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2y + w + 1 \\ y \\ 7y - 3w - 3 \\ w \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 7 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -3 \\ 0 \end{pmatrix}, \quad y, w \in \mathbb{R}$$

(b)

$$\begin{array}{c} \begin{matrix} x & y & z & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & 0 & -1 & 1 \end{array}\right] \end{array} \xrightarrow[-R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -4 & 1 \end{array}\right]$$

$\Rightarrow$ This system is inconsistent. We cannot have

$y-4z=0$ AND $y-4z=1$ because $0 \neq 1$!

$\Rightarrow$ The system has no solution.

3 (a)
$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 4 & 8 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}R_2 \to R_2]{-R_1+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1/4 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-2R_3+R_2 \to R_2]{-3R_3+R_1 \to R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & 2 & 1/4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{array}{ccc} 0 & -1/2 & 1 \\ 2 & 1/4 & -2 \\ -1 & 0 & 1 \end{array}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

(b) The coefficient matrix of the system is the matrix A from part (a).
So the (unique) solution to the system is
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & -1/2 & 1 \\ 2 & 1/4 & -2 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \boxed{\begin{pmatrix} 4\frac{1}{2} \\ -6\frac{1}{4} \\ 3 \end{pmatrix}}$$

④ (a) $z = \sqrt{4-x^2-y^2}$, since $z \geq 1 \Rightarrow z \geq 0$

$r(x,y) = \langle x, y, \sqrt{4-x^2-y^2} \rangle \qquad (x^2+y^2 \leq 3)$

$r_x = \langle 1, 0, \frac{-x}{\sqrt{4-x^2-y^2}} \rangle$
$r_y = \langle 0, 1, \frac{-y}{\sqrt{4-x^2-y^2}} \rangle$
$\Rightarrow r_x \times r_y = \langle \frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1 \rangle$

$$\Rightarrow \|r_x \times r_y\| = \sqrt{\frac{x^2+y^2+4-x^2-y^2}{4-x^2-y^2}} = \frac{2}{\sqrt{4-x^2-y^2}}$$

$$\Rightarrow A(S) = \iint_S dS = \iint \frac{2}{\sqrt{4-x^2-y^2}}\, dA = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2}{\sqrt{4-r^2}}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^{2} u^{-1/2}\, du\, d\theta = 2\pi \cdot 2 \cdot u^{1/2}\Big]_1^2 = \boxed{4\pi(\sqrt{2}-1)}$$

(b) $r(u,v) = \langle u+3, v+2, uv^2 \rangle$

$r_u = \langle 1, 0, v^2 \rangle$
$r_v = \langle 0, 1, 2uv \rangle$
$\Rightarrow r_u \times r_v = \langle -v^2, -2uv, 1 \rangle = n$

$(4,4,4)$ lies on S because $r(\overset{u}{1}, \overset{v}{2}) = \langle 4,4,4 \rangle$ & $n = \langle -4, -4, 1 \rangle$ at this point

So an equation for the tangent plane to S at $(4,4,4)$ is:

$$\boxed{-4(x-4) + (-4)(y-4) + 1\cdot(z-4) = 0}$$

⑤ (a) $\det(A-\lambda I) = \begin{vmatrix} 6-\lambda & 1 \\ -3 & 2-\lambda \end{vmatrix} = (6-\lambda)(2-\lambda)+3 = \lambda^2 - 8\lambda + 15$
$= (\lambda-3)(\lambda-5)$

$\Rightarrow$ eigenvalues of A: 3 and 5

$\underline{\underline{\lambda = 3}}$

$$A - \lambda I = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \xrightarrow{\text{row reduces to}} \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix} \longrightarrow 3x_1 + x_2 = 0$$

$$\Rightarrow x_2 = -3x_1$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -3x_1 \end{pmatrix} \Rightarrow \text{eigenvectors: } \begin{pmatrix} r \\ -3r \end{pmatrix},\ r \in \mathbb{R}$$

$\underline{\underline{\lambda = 5}}$

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \xrightarrow{\text{row reduces to}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \longrightarrow x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} \Rightarrow \text{eigenvectors: } \begin{pmatrix} -s \\ s \end{pmatrix},\ s \in \mathbb{R}$$

(b) coefficient matrix of the system is the matrix A from part (a)

$$\Rightarrow \underline{\text{general solution:}}\quad y = y_I + y_{II} = \begin{pmatrix} re^{3t} \\ -3re^{3t} \end{pmatrix} + \begin{pmatrix} -se^{5t} \\ se^{5t} \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1(t) = re^{3t} - se^{5t} \\ y_2(t) = -3re^{3t} + se^{5t} \end{cases} \Rightarrow \underline{\text{particular solution}}: \quad \begin{aligned} y_1(0) &= r - s = 1 \\ y_2(0) &= -3r + s = 2 \end{aligned} \Rightarrow -2r = 3 \Rightarrow r = -\tfrac{3}{2}\ \&\ s = -\tfrac{5}{2}$$

$$\boxed{\begin{cases} y_1(t) = -\frac{3}{2}e^{3t} + \frac{5}{2}e^{5t} \\ y_2(t) = \frac{9}{2}e^{3t} - \frac{5}{2}e^{5t} \end{cases}}$$

6 (a) $\frac{\partial f}{\partial x} = \frac{\ln y}{2\sqrt{x}} + yz \Rightarrow f(x,y,z) = \sqrt{x}\ln y + xyz + g(y,z)$

(*) $\frac{\partial f}{\partial y} = \frac{\sqrt{x}}{y} + xz$

$\Rightarrow \frac{\partial f}{\partial y} = \frac{\sqrt{x}}{y} + xz + \frac{\partial g}{\partial y}$

$\Rightarrow \frac{\partial g}{\partial y} = 0$ by comparison to (*)

(**) & $\frac{\partial f}{\partial z} = xy$

$\Rightarrow g(y,z) = h(z)$ (because g is constant with respect to y)

$\Rightarrow f(x,y,z) = \sqrt{x}\ln y + xyz + h(z)$

$\Rightarrow \frac{\partial f}{\partial z} = xy + h'(z)$

$\Rightarrow h'(z) = 0$ by comparison to (**)

$\Rightarrow \boxed{f(x,y,z) = \sqrt{x}\ln y + xyz + k}$ ← constant

↑ general potential function

(b) Since part (a) demonstrates that F is conservative (on the portion of $\mathbb{R}^3$ where $y>0$ and $x\geq 0$) we can evaluate $\int_C F\cdot dr$ by letting $k=0$ in our general potential function and calculating $f(4,e^2,2) - f(1,e,1) =$

$$\left[\sqrt{4}\ln(e^2) + 4(e^2)(2)\right] - \left[\sqrt{1}\ln(e) + 1(e)(1)\right]$$

$$= \boxed{3 + 8e^2 - e}$$

(You may evaluate this directly if you like.)

7 (a) $r(\theta, z) = \langle \cos\theta, \sin\theta, z\rangle,\ 0\leq\theta\leq 2\pi,\ 0\leq z\leq 3$

$r_\theta = \langle -\sin\theta, \cos\theta, 0\rangle$

$r_z = \langle 0, 0, 1\rangle$

$\Rightarrow r_\theta \times r_z = \langle \cos\theta, \sin\theta, 0\rangle$

$\Rightarrow \|r_\theta \times r_z\| = 1$

$$\Rightarrow \iint_S z^2\,dS = \iint z^2 (1)\,dA = \int_0^3\int_0^{2\pi} z^2\,d\theta\,dz = 2\pi\cdot\frac{z^3}{3}\Big]_0^3 = \boxed{18\pi}$$

(b) At P, $t=0$. At Q, $t=1$.

So the length of the curve given by $r(t)$ from P to Q is given by $\int_0^1 \|r'(t)\|\,dt$.

$$r(t) = \left\langle \frac{t^2}{2}, \frac{2\sqrt{2}}{3}t^{3/2}, t\right\rangle \Rightarrow r'(t) = \langle t, \sqrt{2}\,t^{1/2}, 1\rangle$$

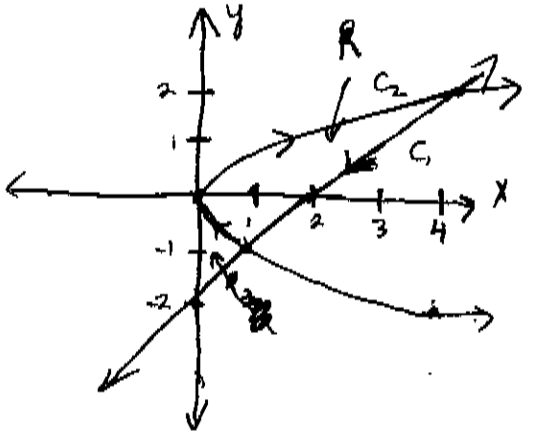
$$\Rightarrow \|r'(t)\| = \sqrt{t^2+2t+1} = \sqrt{(t+1)^2}$$

$= t+1$ BECAUSE $t \geq 0$ so certainly $t \geq -1$

$$\Rightarrow \int_0^1 \|r'(t)\|\,dt = \int_0^1 (t+1)\,dt = \left[\frac{t^2}{2}+t\right]_0^1 = \boxed{\frac{3}{2}}$$

Part II

8



$$F = \langle \underset{P}{-y}, \underset{Q}{x}\rangle$$

(a) C_1: $r(x) = \langle x, x-2\rangle$, $1 \leq x \leq 4$ ("goes 'wrong' way")

$$\int_{C_1} F\cdot dr = -\int_1^4 (2-x)\,dx - \int_1^4 x\,dx$$

$$= \int_1^4 -2\,dx = -2x\Big]_1^4 = \boxed{-6}$$

C_2: $r(y) = \langle y^2, y\rangle$, $-1 \leq y \leq 2$ ("goes 'right' way")

$$\int_{C_2} F\cdot dr = \int_{-1}^2 (-y)(2y\,dy) + \int_{-1}^2 y^2\,dy$$

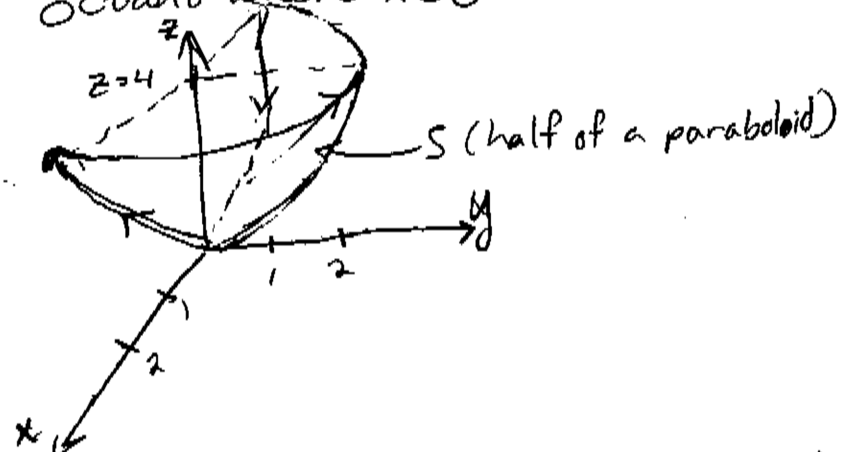
$$= \int_{-1}^2 -y^2\,dy = -\frac{1}{3}y^3\Big]_{-1}^2 = \boxed{-3}$$

$$\Rightarrow \int_C F\cdot dr = (-6)+(-3) = \boxed{-9}$$

(b) $\frac{\partial Q}{\partial x} = 1$, $\frac{\partial P}{\partial y} = -1 \Rightarrow \int_C F\cdot dr = -\iint_R 2\,dA = -2\int_{-1}^2\int_{y^2}^{y+2} dx\,dy$

$$= -2\int_{-1}^2 (y+2-y^2)\,dy = 2\left[\frac{1}{3}y^3 - \frac{y^2}{2} - 2y\right]_{-1}^2 = 2\left(-\frac{10}{3} - \frac{7}{6}\right) = \boxed{-9}$$

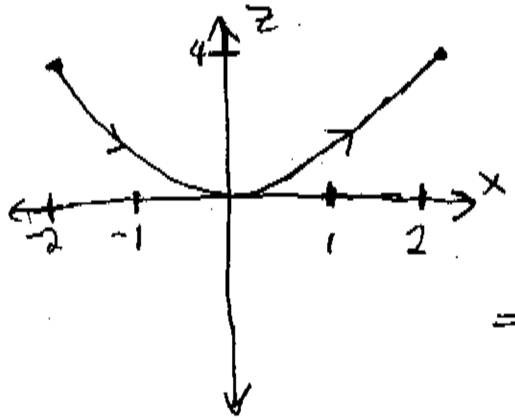
(9) Let's orient C in such a way that the portion visible looking down at the xy-plane "moves" from the octant where $x \geq 0$ to the octant where $x \leq 0$ (counter-clockwise motion)



$F = \langle y, z, x \rangle$

(a) The boundary of S can be broken down into two smooth pieces, one which lives in the xz-plane and the other which lives in the plane $z=4$. We will call these two pieces C_1 and C_2 respectively.

$\underline{\underline{C_1}}$:



$r(x) = \langle x, 0, x^2 \rangle, \ -2 \leq x \leq 2$

$\Rightarrow F(r(x)) = \langle 0, x^2, x \rangle$

& $r'(x) = \langle 1, 0, 2x \rangle$

$\Rightarrow F(r(x)) \cdot r'(x) = 2x^2$

$$\int_{C_1} F \cdot dr = \int_{-2}^{2} F(r(x)) \cdot r'(x)\, dx = \int_{-2}^{2} 2x^2\, dx = \frac{2}{3} x^3 \Big]_{-2}^{2} = \frac{32}{3}$$

$\underline{\underline{C_2}}$: $r(\theta) = \langle 2\cos\theta, 2\sin\theta, 4 \rangle, \ 0 \leq \theta \leq \pi$

$\Rightarrow F(r(\theta)) = \langle 2\sin\theta, 4, 2\cos\theta \rangle$ & $r'(\theta) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$

$\Rightarrow F(r(\theta)) \cdot r'(\theta) = -4\sin^2\theta + 8\cos\theta$

$$\Rightarrow \int_{C_2} F \cdot dr = \int_0^{\pi} (-4\sin^2\theta + 8\cos\theta)\, d\theta = -4\int_0^{\pi} \frac{1-\cos(2\theta)}{2}\, d\theta = -2\pi$$

$$\Rightarrow \int_C F \cdot dr = \boxed{\frac{32}{3} - 2\pi}$$

(6) $\int_C F\cdot dr = \iint_S \operatorname{curl} F \cdot dS$
$\uparrow$
Stokes'

<u>Parametrization of S:</u>

$r(x,y) = \langle x, y, x^2+y^2\rangle \quad (x^2+y^2 \le 4)$

$r_x = \langle 1, 0, 2x\rangle$
$r_y = \langle 0, 1, 2y\rangle$ $\Rightarrow r_x \times r_y = \langle -2x, -2y, 1\rangle = n$

(Use the "right-hand rule" to see that this choice of normal is consistent with the orientation of the boundary of S)

$$\operatorname{curl} F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \langle -1, -1, -1\rangle$$

$\Rightarrow \operatorname{curl} F(r(x,y)) = \langle -1, -1, -1\rangle$

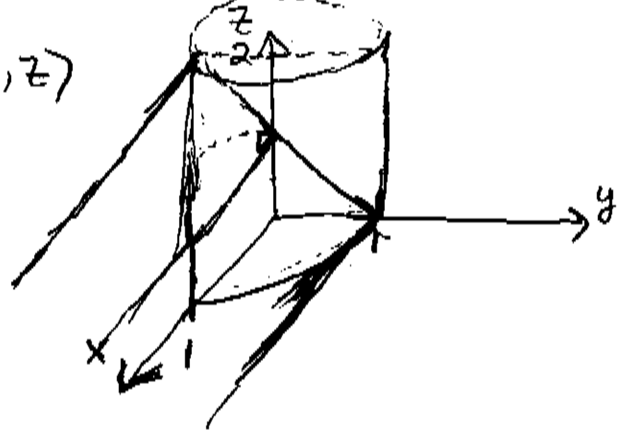
$\Rightarrow \operatorname{curl} F(r(x,y))\cdot n = 2x+2y-1$

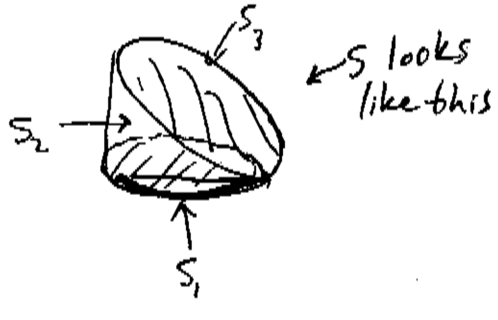
$$\Rightarrow \iint_S \operatorname{curl} F\cdot dS = \iint (2x+2y-1)\,dA = \int_0^{2\pi}\int_0^2 (2r\cos\theta + 2r\sin\theta - 1)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\left(2\sin\theta\left[\frac{r^3}{3}\right]_0^2 - \left[\frac{r^2}{2}\right]_0^2\right)d\theta$$

$$= -\frac{16}{3}(\cos\theta)_0^{\pi} - 2\pi = \boxed{\frac{32}{3} - 2\pi}$$

(10) $F = \langle x, y, z\rangle$





(a) We have to evaluate $\iint_S F\cdot dS$ in three pieces.

$\iint_{S_1} F\cdot dS_1$ Parametrization of S_1:

$r(r,\theta)=\langle r\cos\theta, r\sin\theta, 0\rangle$, $0\le r\le 1$, $0\le\theta\le 2\pi$

$r_r=\langle \cos\theta, \sin\theta, 0\rangle$
$r_\theta=\langle -r\sin\theta, r\cos\theta, 0\rangle$ $\Rightarrow r_r\times r_\theta=\langle 0,0,r\rangle$

$\Rightarrow r_\theta\times r_r=\langle 0,0,-r\rangle$ is an outward-pointing normal

$F=F(r(r,\theta))=\langle r\cos\theta, r\sin\theta, 0\rangle \Rightarrow F\cdot n=0 \Rightarrow \iint_{S_1} F\cdot dS_1=0$

$\iint_{S_2} F\cdot dS_2$ Parametrization of S_2:

$r(\theta,z)=\langle\cos\theta, \sin\theta, z\rangle$, $0\le\theta\le 2\pi$, $0\le z\le 1-\cos\theta$

$\Rightarrow r_\theta=\langle -\sin\theta, \cos\theta, 0\rangle$
& $r_z=\langle 0,0,1\rangle$ $\Rightarrow r_\theta\times r_z=\langle\cos\theta, \sin\theta, 0\rangle$
(Verify that this points outward.)

$F=F(r(\theta,z))=\langle\cos\theta, \sin\theta, z\rangle \Rightarrow F\cdot n=\cos^2\theta+\sin^2\theta=1$

$\Rightarrow \iint_{S_2} F\cdot dS_2=\iint 1\,dA=\int_0^{2\pi}\int_0^{1-\cos\theta} dz\,d\theta=\int_0^{2\pi}(1-\cos\theta)d\theta=2\pi$

$\iint_{S_3} F\cdot dS_3$ Parametrization of S_3:

$r(x,y)=\langle x, y, 1-y\rangle$ (without limits)

$r_x=\langle 1,0,0\rangle$
$r_y=\langle 0,1,-1\rangle$ $\Rightarrow r_x\times r_y=\langle 0,1,1\rangle$ ← points upward; could have obtained this from equation for plane $y+z=1$

$F = F(r(x,y)) = \langle x, y, 1-y \rangle$

$\Rightarrow F \cdot n = 0 + y + 1 - y = 1$

$\Rightarrow \iint_{S_3} F \cdot dS_3 = \iint 1\, dA = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \pi$

$\iint_S F \cdot dS = 0 + 2\pi + \pi = \boxed{3\pi}$

(b) $\operatorname{div} F = 1 + 1 + 1 = 3$

$$\iint_S F \cdot dS \underset{\substack{\uparrow \\ \text{Divergence} \\ \text{Theorem}}}{=} \iiint_T \overset{\operatorname{div} F}{3}\, dV = 3\int_0^{2\pi}\int_0^1\int_0^{1-\cos\theta} r\,dz\,dr\,d\theta$$

$$= \frac{3}{2}\int_0^{2\pi}(1-\cos\theta)\,d\theta = \frac{3}{2}(2\pi) = \boxed{3\pi}$$