THE CITY COLLEGE **DEPARTMENT OF MATHEMATICS**

Mathematics 39200 **Final Examination** **December 2006**

INSTRUCTIONS: Answer five questions from Part I and two questions from Part II. Show all work. Calculators are not permitted.

PART I. Answer five complete questions from this part. (14 points each)

1. Let A be the matrix $\begin{pmatrix} 2 & 0 & 2 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{pmatrix}$.

a) Find A^{-1}. Use the method of your choice.

b) Use your answer in part a) to solve $\begin{cases} 2x & & +2z & =2 \\ 2x & +y & +z & =5 \\ 3x & +2y & +2z & =8 \end{cases}$.

c) Solve the system in b) for x (not y or z) by using Cramer's Rule.

2. Solve the simultaneous differential equations $\begin{cases} y'_1 = 3y_1(t) + y_2(t) \\ y'_2 = y_1(t) + 3y_2(t) \end{cases}$ for $y_1(t)$ and $y_2(t)$ subject to initial conditions $y_1(0) = 1$ and $y_2(0) = 2$. First find eigenvalues and eigenvectors of an appropriate matrix. No credit for any other method!

3. Use Gaussian Elimination to solve the following linear systems:

a) $\begin{cases} w+x+y+z=1 \\ 2w \quad +2y \quad =4 \\ \qquad y+z=6 \end{cases}$ **b)** $\begin{cases} w+x+y+z=1 \\ 2w \quad +2y \quad =4 \\ 3w+x+3y+z=6 \end{cases}$ c) $\begin{cases} 2x & & +2z & =2 \\ 2x & +y & +z & =5 \\ 3x & +2y & +2z & =8 \end{cases}$

4. Let $\vec{F}(x,y,z) = < ye^z + y, xe^z + x + 1, xye^z + 1 >$

a) Find a potential function $f(x,y,z)$ with $\nabla f = \vec{F}$

b) Let C be the straight line segment joining (1,1,1) to (2,2,0).
Use the result of a) to evaluate $\int_C \vec{F} \cdot d\vec{r}$

5. Let T be the solid ball $x^2 + y^2 + z^2 \le 4$. Let T_1 be the part of T that lies above the cone $z = \sqrt{x^2 + y^2}$. Let T_2 be the part of T with $x \le 0$ and $z \le 0$. Use spherical co-ordinates (ρ, ϕ, θ) to set up bounds of integration for **a)** $\iiint_{T_1} (x^2 + y^2 + z^2)\,dV$ and **b)** $\iiint_{T_2} (x^2 + y^2 + z^2)\,dV$

Then evaluate *one* of the definite integrals **a)** or **b)**.

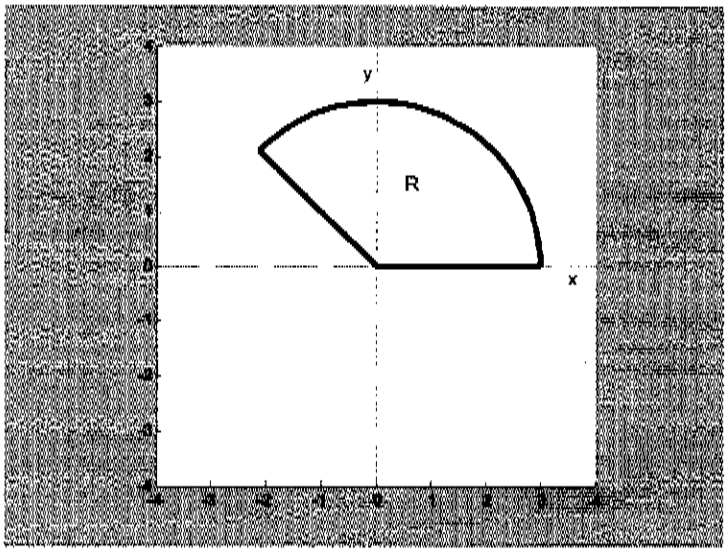
Please turn the page for the rest of Part I and for Part II.

6. **a)** Find the surface area of the part of the surface S: $z = x^2 + y^2$ with $1 \le z \le 4$.
b) Find an equation of the tangent plane to S at the point $(1,1,2)$.

End of Part I. Make sure you answered five complete questions from this part.

PART II: Answer 2 complete questions from this part (15 points each).

7. Let R be the region shown below bounded by the line $y = -x$, the circle $x^2 + y^2 = 9$, and the line $y = 0$.



Suppose the boundary C of R is oriented counterclockwise.

Evaluate $\int_C -y\,dx + x\,dy$

a) directly, as a line integral, *and*
b) as a double integral, by using Green's Theorem.

8. Let S be that part of the surface $z = 1 - x^2$ in the first octant with $0 \le y \le 2$. Let C be the boundary of S, oriented counterclockwise when viewed from above.
If $\vec{F} = \langle 1, 0, y^2 \rangle$, Calculate $\int_C \vec{F} \cdot d\vec{r}$

a) directly as a line integral, *and*
b) as a surface integral, by using Stokes' Theorem.

9. Let T be the solid bounded below by $z = x^2 + y^2$ and above by $z = 4$, and let S be the boundary surface of T, with outward pointing unit normal vector.
Let $\vec{F}$ be the vector field $< x, y, 1 >$.
Calculate $\iint_S \vec{F} \cdot d\vec{S}$

a) directly as a surface integral, *and*
b) as a triple integral, by using the Divergence Theorem.

END OF EXAM. Make sure that you answered five complete questions from Part I and two complete questions from Part II.

Math 39200, Fall 2006 Final Exam Solutions

Part I

1. (a)

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-3)R_1+R_3 \to R_3]{(-2)R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -3/2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-2)R_2+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_3+R_2 \to R_2]{-R_3+R_1 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -1/2 & -1 & 1 \\ 0 & 0 & 1 & 1/2 & -2 & 1 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 0 & 2 & -1 \\ -1/2 & -1 & 1 \\ 1/2 & -2 & 1 \end{bmatrix}}_{A^{-1}}$$

(b) $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -1 \\ -1/2 & -1 & 1 \\ 1/2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}}$$

(c) <u>Cramer's Rule:</u>

If A is invertible, $$x = \frac{\begin{vmatrix} 2 & 0 & 2 \\ 5 & 1 & 1 \\ 8 & 2 & 2 \end{vmatrix}}{\det(A)} = \frac{4}{2} = \boxed{2}$$

Numerator: $2((2)-2)+2(10-8)=4$

Denominator: $2(2-2)+2(4-3)=2$

(2) $$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2-6\lambda+8 = (\lambda-2)(\lambda-4)$$

$\Rightarrow$ Eigenvalues: 2 & 4

<u>$\lambda=2$</u>

$$A-\lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1+x_2=0$$

$$x_1=-x_2$$

$\Rightarrow$ eigenvectors $r\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -r \\ r \end{pmatrix}$, $r\in\mathbb{R} \Rightarrow y_I = \begin{pmatrix} -re^{2t} \\ re^{2t} \end{pmatrix}$

$\underline{\lambda = 4}$

$$A - \lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{row reduces to}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 - x_2 = 0$$
$$\Rightarrow x_1 = x_2$$

$$\Rightarrow \text{eigenvectors } \begin{pmatrix} s \\ s \end{pmatrix},\ s \in \mathbb{R} \Rightarrow y_{II} = \begin{pmatrix} se^{4t} \\ se^{4t} \end{pmatrix}$$

$$\Rightarrow y = y_I + y_{II} = \begin{pmatrix} -re^{2t} + se^{4t} \\ re^{2t} + se^{4t} \end{pmatrix} \begin{matrix} \leftarrow y_1(t) \\ \leftarrow y_2(t) \end{matrix}$$

$$\left.\begin{aligned} y_1(0) &= -r + s = 1 \\ y_2(0) &= r + s = 2 \end{aligned}\right\} \Rightarrow 2s = 3 \Rightarrow s = \frac{3}{2} \ \& \ r = \frac{1}{2}$$

$$\boxed{\text{particular solution} \begin{cases} y_1(t) = -\frac{1}{2}e^{2t} + \frac{3}{2}e^{4t} \\ y_2(t) = \frac{1}{2}e^{2t} + \frac{3}{2}e^{4t} \end{cases}}$$

③ (a)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 2 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right] \xrightarrow{-R_3 + R_1 \to R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_1 \to R_1} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -4 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 6 \end{array}\right] \end{array} \Rightarrow \begin{aligned} w - z &= -4 \\ x + z &= -1 \\ y + z &= 6 \end{aligned} \Rightarrow \begin{aligned} w &= z - 4 \\ x &= -z - 1 \\ y &= -z + 6 \end{aligned}$$

general solution: $\begin{pmatrix} z-4 \\ -z-1 \\ -z+6 \\ z \end{pmatrix}$, $z \in \mathbb{R}$ ← free variable

(b) $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 & 4 \\ 3 & 1 & 3 & 1 & 6 \end{array}\right]$

$$\xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \begin{array}{ccccc} w & x & y & z & \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 & 2 \\ 0 & -2 & 0 & -2 & 3 \end{array}\right]$$

$\Rightarrow$ system is inconsistent, as we cannot have $-2x-2z=2$ **AND** $-2x-2z=3$ because $2 \neq 3$

(c) $\begin{array}{ccc} x & y & z \end{array}$ $\left[\begin{array}{ccc|c} 2 & 0 & 2 & 2 \\ 2 & 1 & 1 & 5 \\ 3 & 2 & 2 & 8 \end{array}\right] \xrightarrow{\frac{1}{2}R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 5 \\ 3 & 2 & 2 & 8 \end{array}\right]$

$$\xrightarrow[-3R_1+R_3\to R_3]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 2 & -1 & 5 \end{array}\right] \xrightarrow{-2R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_3+R_2\to R_2]{-R_3+R_1\to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \Longrightarrow \text{solution } \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \text{ or } \boxed{x=2,\ y=2,\ \text{and } z=-1}$$

④ (a) $\frac{\partial f}{\partial x} = ye^z + y \Rightarrow f(x,y,z) = \int (ye^z + y)\,dx + g(y,z)$

$\frac{\partial f}{\partial y} = xe^z + x + 1$ (*) $\Rightarrow f(x,y,z) = xye^z + xy + g(y,z)$

$\Rightarrow \frac{\partial f}{\partial y} = xe^z + x + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = 1$ (*)

$\frac{\partial f}{\partial z} = xye^z + 1$ (**) $\Rightarrow g(y,z) = y + h(z)$

$\Rightarrow f(x,y,z) = xye^z + xy + y + h(z)$

$\Rightarrow \frac{\partial f}{\partial z} = xye^z + h'(z) \Rightarrow h'(z) = 1$

general potential function

$\Rightarrow h(z) = z + k$, k a constant

$$\boxed{f(x,y,z) = xye^z + xy + y + z + k}$$

(b) By the fundamental theorem for line integrals, we have $\int_C F \cdot dr = f(r(b)) - f(r(a))$

where $\nabla f = F$ and C starts at $r(a)$ and ends at $r(b)$

As posed, the problem is unclear about the initial and terminal points. Let us assume the initial point is $(1,1,1)$ and the terminal point is $(2,2,0)$

Then $\int_C F \cdot dr = f((2,2,0)) - f((1,1,1))$ (Let $K=0$ in solution to (a).)

$= (4+4+2+0) - (e+1+1+1)$

$= \boxed{7 - e}$

(5)

(a) $$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{2} \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{32}{5} \cdot 2\pi \left(-\cos\phi\right)_0^{\pi/4} = \frac{64\pi}{5}\left(\frac{2-\sqrt{2}}{2}\right) = \boxed{\frac{32\pi(2-\sqrt{2})}{5}}$$

(b) $$\int_{\pi/2}^{\pi}\int_{\pi/2}^{3\pi/2}\int_0^{2} \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \frac{32}{5} \cdot (\pi) \cdot \left(-\cos\phi\right)_{\pi/2}^{\pi} = \boxed{-\frac{32\pi}{5}}$$

(You were only asked to evaluate one. I evaluated both above. The crucial step is setting up these integrals.)

(6) (a) $z = x^2 + y^2$

$r(x,y) = \langle x, y, x^2+y^2 \rangle$

$r_x = \langle 1, 0, 2x \rangle$

$r_y = \langle 0, 1, 2y \rangle$ $\Rightarrow r_x \times r_y = \langle -2x, -2y, 1 \rangle$

$\|r_x \times r_y\| = \sqrt{4x^2+4y^2+1}$

$$\Rightarrow A(S) = \iint \sqrt{4x^2+4y^2+1} \, dA = \int_0^{2\pi}\int_1^{2} r\sqrt{4r^2+1} \, dr \, d\theta$$

$$= 2\pi\left(\frac{1}{8}\right)\int_5^{17} u^{1/2} \, du = \frac{\pi}{4} \cdot \frac{2}{3} \cdot \left(17^{3/2} - 5^{3/2}\right) = \boxed{\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)}$$

(b) At $(1,1,2)$, $x=y=1$

$\Rightarrow r_x \times r_y = \langle -2,-2,1\rangle$

<u>Tangent plane</u> has eqn:

$$\boxed{(-2)(x-1)+(-2)(y-1)+1(z-2)=0}$$

$\boxed{\text{Part II}}$

(7) (a) C has three parts to it

<u>C_1:</u> start at $(0,0)$ & end at $(3,0)$

$r(t)=\langle 3t,0\rangle$, $0\le t\le 1$

$$\int_{C_1} -y\,dx + x\,dy = \int_0^1 0(3dt) + \int_0^1 3t(0dt) = 0$$

<u>C_2:</u> Start at $(3,0)$ & go to? $x^2+y^2=9$ & $y=-x$

$\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ $\Rightarrow 2x^2=9 \Rightarrow x^2=\frac{9}{2} \Rightarrow x=-\frac{3\sqrt{2}}{2}$

& $y=\frac{3\sqrt{2}}{2}$

$r(\theta)=\langle 3\cos\theta, 3\sin\theta\rangle$, $0\le\theta\le\frac{3\pi}{4}$

$$\int_{C_2} y\,dx + x\,dy = \int_0^{3\pi/4} -3\sin\theta(-3\sin\theta\,d\theta) + \int_0^{3\pi/4} 3\cos\theta(3\cos\theta\,d\theta)$$

$$= 9\int_0^{3\pi/4} 1\,d\theta \quad \boxed{\frac{27\pi}{4}}$$

<u>C_3:</u> start at $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ and go to $(0,0)$ along $y=-x$

$r(x)=\langle x,-x\rangle$, $0\ge x\ge -\frac{3\sqrt{2}}{2}$ goes in "right" direction

$$\int_{C_3} -y\,dx + x\,dy = \int_{-\frac{3\sqrt{2}}{2}}^{0} +x\,dx - x\,dx = 0$$

$$\Rightarrow \int_C -y\,dx + x\,dy = 0 + \left(\frac{27\pi}{4}\right) + 0 = \boxed{\frac{27\pi}{4}}$$

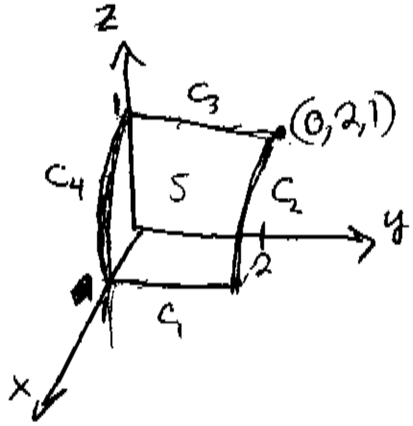
)) $$\int_C F\cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Here, $P = -y$ & $Q = x \Rightarrow \frac{\partial P}{\partial y} = -1$ & $\frac{\partial Q}{\partial x} = 1$

$$\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1-(-1) = 2$$

$$2\iint_D dA = 2\underbrace{\left(\tfrac{3}{8}\right)9\pi}_{\text{area of region}} = \boxed{\frac{27\pi}{4}}$$

area of region



(8) $F = \langle 1, 0, y^2\rangle$

(a) C_1:

$r(t) = \langle 1, 2t, 0\rangle$

$r'(t) = \langle 0, 2, 0\rangle$

& $F(r(t)) = \langle 1, 0, 4t^2\rangle$

$\Rightarrow F(r(t))\cdot r'(t) = 0 \Rightarrow$ no contribution to line integral here

C_2:

$r(x) = \langle x, 2, 1-x^2\rangle$, $0 \le x \le 1$ (goes in "wrong" direction)

$\Rightarrow F(r(x)) = \langle 1, 0, 4\rangle$ & $r'(x) = \langle 1, 0, -2x\rangle$

$$\Rightarrow F(r(x))\cdot r'(x) = 1-8x \Rightarrow \int_{C_2} F\cdot dr = -\int_0^1 (1-8x)\,dx = [x-4x^2]_0^1$$

$= +3$

C_3:

$r(t) = \langle 0,2,1\rangle(1-t) + \langle 0,0,1\rangle t = \langle 0, 2-2t, 1\rangle$, $0\le t\le 1$

$\Rightarrow F(r(t)) = \langle 1, 0, (2-2t)^2\rangle$ & $r'(t) = \langle 0,-2,0\rangle$

$F(r(t))\cdot r'(t) = 0 \Rightarrow$ no contribution from C_3

$\underline{C_4:}$

$r(x) = \langle x, 0, 1-x^2 \rangle$, $0 \le x \le 1$ (goes in "~~right~~" direction)

$$\int_{C_4} F \cdot dr = \int_0^1 F(r(x)) \cdot r'(x)\, dx$$

$F(r(x)) = \langle 1, 0, 0 \rangle$ & $r'(x) = \langle 1, 0, -2x \rangle$

$$= \int_0^1 1\, dx = +1$$

$$\int_C F \cdot dr = 0 + 3 + 0 + +1 = \boxed{4}$$

(b) $\int_C F \cdot dr = \iint_S \text{curl} F \cdot dS$, where C is the boundary of S, both positively oriented

$r(x,y) = \langle x, y, 1-x^2 \rangle$, $0 \le x \le 1$, $0 \le y \le 2$

$r_x = \langle 1, 0, -2x \rangle$

$r_y = \langle 0, 1, 0 \rangle$

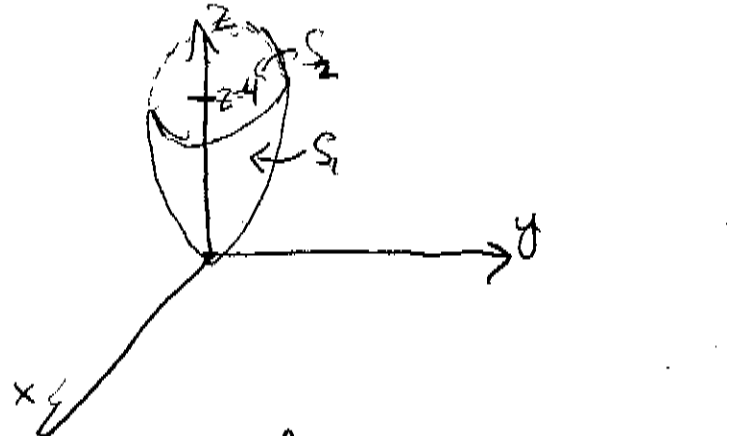
$\Rightarrow r_x \times r_y = \underbrace{\langle 2x, 0, 1 \rangle}_{=n}$ ← points upward as k-component > 0

$$\text{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 0 & y^2 \end{vmatrix} = \langle 2y, 0, 0 \rangle$$

$\text{curl} F(r(x,y)) = \langle 2y, 0, 0 \rangle \Rightarrow \text{curl} F \cdot n = 4xy$

$$\Rightarrow \int_C F \cdot dr = \int_0^2 \int_0^1 4xy\, dx\, dy = \int_0^2 2x^2 y \Big]_0^1 dy = \int_0^2 2y\, dy = y^2 \Big]_0^2 = \boxed{4}$$

(9)



$F = \langle x, y, 1 \rangle$

(a) S is a closed surface made up of two pieces, as indicated above. We will need the outward-pointing normal on each piece.

Parametrization of S_1:

$r(x,y) = \langle x, y, x^2+y^2 \rangle \Rightarrow F(r(x,y)) = \langle x, y, 1 \rangle$

$r_x = \langle 1, 0, 2x \rangle$
$r_y = \langle 0, 1, 2y \rangle$ $\Rightarrow r_x \times r_y = \langle -2x, -2y, 1 \rangle$ ← points inward

$\Rightarrow n = r_y \times r_x$ is the outward-pointing normal

$$\iint_{S_1} F \cdot dS_1 = \iint \langle x, y, 1 \rangle \cdot \langle 2x, 2y, -1 \rangle \, dA$$

$$= \iint (2x^2+2y^2-1)\, dA = \int_0^{2\pi}\int_0^2 (2r^2-1)\, r\, dr\, d\theta$$

$$= 2\pi \cdot \left[\frac{r^4}{2} - \frac{r^2}{2}\right]_0^2 = 2\pi(6) = \boxed{12\pi}$$

Parametrization of S_2:

$r(x,y) = \langle x, y, 4 \rangle \Rightarrow F(r(x,y)) = \langle x, y, 1 \rangle$

$r_x = \langle 1, 0, 0 \rangle$
$r_y = \langle 0, 1, 0 \rangle$ $\Rightarrow$ upward-pointing normal $= \langle 0, 0, 1 \rangle = n$
outward

$$\iint_{S_2} F \cdot dS_2 = \int_0^{2\pi}\int_0^2 1 \cdot r\, dr\, d\theta = \boxed{4\pi} \Rightarrow \iint_S F \cdot dS = 12\pi + 4\pi = \boxed{16\pi}$$

(b) $\operatorname{div} F = 1 + 1 + 0 = 2$

$$\Rightarrow \iiint_E \operatorname{div} F \, dV = \int_0^{2\pi} \int_0^{2} \int_{r^2}^{4} 2 \cdot r \, dz \, dr \, d\theta$$

$\leftarrow$ up to plane

$\leftarrow$ from paraboloid

$$= 4\pi \int_0^2 (4r - r^3) \, dr$$

$$= 4\pi \left[2r^2 - \frac{r^4}{4} \right]_0^2 = 4\pi(4) = \boxed{16\pi}$$