MATH 203 Final Exam, December 2011

Instructions: Complete every question in Part I (Questions 1-7.) Complete three of the five questions in Part II (Questions 8-12). Each question is worth 10 points.

No calculators or other electronic devices may be used during the examination. Answers may be left in terms of $\sqrt{7}$, $\sin(3)$, π, etc. if needed. Give reasons for all answers. This exam will last 2 hours and 15 minutes. Good luck!

1. (a) Find an equation for the plane through the points $A = (0, 1, 2)$, $B = (1, 0, -1)$, and $C = (1, 2, 3)$.

   (b) Find parametric equations for the line through $(4, 1, 3)$ and perpendicular to the plane found in (a) above.

   (c) Let $\vec{v} = \langle 0, 2, -2 \rangle$ Is the line found in (b) perpendicular to $\vec{v}$, parallel to $\vec{v}$ or neither of these?

2. At points (x, y, z) in a region of space for which $x^2 + y^2 \geq 1$ and $z \geq 0$, there is an electric charge $E(x, y, z) = z + z\ln(x^2 + y^2)$.

   (a) Find the rate at which the electric charge is changing at $(1, 0, 2)$ in the direction towards the point $(4, 4, 7)$.

   (b) Find the direction of greatest increase in E at $(1, 0, 2)$.

   (c) At each point (s, t) on the ground in a physics lab, the electric charge at position $(x, y, z) = (s + t, s - t, 2st)$ is measured. Find the rate, $\frac{\partial E}{\partial s}$, at which the electric charge is changing with respect to s at the point $(s, t) = (1, 1)$.

3. Find all local maxima and minima and all saddle points of the function $f(x, y) = 3x^2 - 12xy + 8y^3$.

4. (a) Use polar coordinates to evaluate $\int\int_R \sqrt{x^2 + y^2}\, dA$, where R is the region bounded by $x^2 + y^2 = 2x$, $y = x$ and $y = -x$.

   (b) Find an equation of the tangent plane to the surface $z = x(y^2 + 2)$ at $(3, -1, 9)$.

5. Find the volume of the region bounded by

   $$y = x^2, \quad y = 4, \quad x = 0, \quad x = 1, \quad z = x, \quad z = x + y.$$

6. For each of the following series, state whether they are absolutely convergent, conditionally convergent, or divergent. Name a test which supports each conclusion and show the work to apply the test.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{2 + (-1)^n}{n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$

7. Find the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{(n+3)6^n}$. (Remember to check the endpoints, if applicable.)

Part II: (30 points) Solve 3 complete problems out of 5

8. (a) Find the z-coordinate of the center of mass of a solid bounded below by the xy-plane, on the inside by the cone $z^2 = x^2 + y^2$ and on the outside by the sphere $x^2 + y^2 + z^2 = 4$ and having density $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

   (b) Find the limit or show it does not exist. Justify your answer:

   $$\lim_{(x,y)\to(0,0)} \frac{2x\sin y}{x^2 + y^2}$$

9. a) Let $f(x, y) = e^{3x-y}\cos(x - 1)$. Estimate $f(.98, 3.01)$ using differentials (that is, with a linear approximation).

   (b) Find an equation of the tangent plane to the level surface

   $2\sin(x - y) + e^{4y^2 - z^2} = 1$ at $(1, 1, 2)$.

10. (a) Find the surface area of the portion of the surface $z = x^2 - y^2$ which is inside the cylinder $x^2 + y^2 = 3$.

    (b) Sketch the graph of $4x - x^2 + 4y^2 + 4z^2 = 0$, labelling the coordinates of any vertices.

11. Find the mass of the region, described in cylindrical coordinates (r, θ, z), which is bounded around the sides by $r = 2 + 2\cos\theta$, on top by $z = 2$ and on the bottom by $z = 0$, given that the density at any point (r, θ, z) is equal to z.

12. A student slightly incorrectly calculates the Maclaurin series for $f(x) = e^x + e^{-2x}$ as $\sum_{n=0}^{\infty} \frac{1 - 2^n}{n!} x^n$.

    (a) Find the correct Maclaurin series representation for $f(x)$

    (b) Estimate $f(1/4)$ with an error of at most one tenth (justify that the error in your estimate is at most .1). The answer may be expressed as a sum of unsimplified fractions.

    (c) Find the first three terms of the Maclaurin series for $\frac{df}{dx}$.