# CALCULUS III
# THE CHAIN RULE, DIRECTIONAL DERIVATIVES, AND GRADIENT

**MATH 20300 DD & ST2**

**prepared by**

**Antony Foster**
**Department of Mathematics (office: NAC 6-273)**
**The City College of The City University of New York**
**Convent Avenue At 138th Street**
**New York, NY 10031**

afoster00@ccny.cuny.edu
afoster1955@gmail.com

# Contents

# The Chain Rule

In this discussion we will derive versions of the chain rule for functions of two or three real variables. These new versions will allow us to generate useful relationships among the drivatives and partial derivatives of various functions.

The chain rule for derivatives (you learned this in Calculus).

If $y = f(x)$ is a differentiable function of a single variable $x$ and $x = g(t)$, is also a differentiable function of a single variable $t$, then the chain rule for functions of a single variable states that, under composition, $y = (f \circ g)(t)$, $y$ becomes a differentiable function of $t$ with

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \text{ or } f'(t) = f'(g(t))\,g'(t).$$

We will now derive a version of the chain rule for functions of two variables.

Suppose the situation is that $z = f(x, y)$ is a function of $x$ and $y$, and suppose that $x$ and $y$ are in turn functions of a single variable $t$, say

$$x = x(t), \quad y = y(t).$$

The composition $z = f(x(t), y(t))$ then expresses $z$ as a function of the single variable $t$. Thus, it makes sense to ask for the derivative $dz/dt$ and we can inquire about its relationship to the derivatives $\partial z/\partial x$, $\partial z/\partial y$, $dx/dt$, and $dy/dt$. Letting $\Delta x$, $\Delta y$, and $\Delta z$ denote the changes in $x$, $y$, and $z$, respectively, that correspond to a change of $\Delta t$ in $t$, we have

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t}, \quad \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}, \text{ and } \frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}.$$

It follows from our discussions about approximations that

$$\Delta z \approx \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \tag{1}$$

where the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are evaluate at $(x(t), y(t))$. Dividing both sides of (1) by $\Delta t$ yields

$$\frac{\Delta z}{\Delta t} \approx \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t}. \tag{2}$$

Taking the limit as $\Delta t \to 0$ of both sides of (2) suggests the following result.

**Theorem 0.1** [Two-Variable Chain Rule]. If $x = x(t)$ and $y = y(t)$ are differentiable at $t$, and if $z = f(x, y)$ is differentiable at the point $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ is differentiable at $t$ and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{3}$$

where the ordinary derivatives are evaluated at $t$ and the partial derivatives are evaluated at $(x, y)$.

**Example 0.1** Suppose that

$$z = x^2 y, \quad x = t^2, \quad y = t^2.$$

Use the chain rule to find $dz/dt$, and check the result by expressing $z$ as a function of $t$ and differentiating directly.

**Solution** By the chain rule

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy)(2t) + (x^2)(3t^2) \\ &= (2t^5)(2t) + (t^4)(3t^2) = 7t^6. \end{aligned}$$

Alternatively, we can express $z$ directly as a function of $t$,

$$z = x^2 y = (t^2)^2(t^3) = t^7$$

and then differentiate to obtain $dz/dt = 7t^6$. However, this procedure may not allways be convenient.

**Example 0.2** Suppose that

$$z = \sqrt{xy + y}, \quad x = \cos\theta, \quad y = \sin\theta$$

Use the chain rule to find $dz/d\theta$ when $\theta = \pi/2$.

**Solution** From the chain rule with $\theta$ in place of $t$,

$$\frac{dz}{d\theta} = \frac{\partial z}{\partial x}\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\frac{dy}{d\theta}$$

we obtain

$$\frac{dz}{d\theta} = \frac{1}{2}(xy + y)^{-1/2}(y)(-\sin\theta) + \frac{1}{2}(xy + y)^{-1/2}(x + 1)(\cos\theta).$$

When $\theta = \pi/2$, we have

$$x = \cos\frac{\pi}{2} = 0, \quad y = \sin\frac{\pi}{2} = 1.$$

Substituting $x = 0$ $y = 1$, $\theta = \pi/2$ in the formula for $dz/d\theta$ yields

$$\frac{dz}{d\theta}\Big|_{\theta=\pi/2} = \frac{1}{2}(1)(1)(-1) + \frac{1}{2}(1)(1)(0) = -\frac{1}{2}.$$

Theorem 0.1 has a natural extension to functions $w = f(x, y, z)$ of three variables which we state without proof

**Theorem 0.2** [Three-Variable Chain Rule] If each of the functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ is differentiable at $t$, and if $w = f(x, y, z)$ is differentiable at the point $(x, y, z) = (x(t), y(t), z(t))$, then $w = f(x(t), y(t), z(t))$ is differentiable at $t$ and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \tag{4}$$

where the ordinary derivatives are evalated at $t$ and the partial derivatives are evaluated at $(x, y, z)$.

One of the principal uses of the chain rule for functions of a single variable was to compute formulas for the derivatives of compositions of functions. Theorem 0.1 and Theorem 0.2 are important not so much for the computation of formulas but because they allow us to express relationships among various derivatives. As illustrations, we revisit the topics of implicit differentiation and related rates problems.

Implicit Differentiation (A revisit)

Consider the special case where $z = f(x, y)$ is a function of $x$ and $y$ and $y$ is a differentiable function of $x$. Equation (3) then becomes

$$\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \tag{5}$$

This result can be used to find derivaties of functions that are defined implicitly. For example, suppose that the equation

$$f(x, y) = 0 \tag{6}$$

defines $y$ implicitly as a differentiable function of $x$ and we are interested in finding $dy/dx$. Differentiating both sides of (6) with respect to $x$ and applying (5) yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0.$$

Thus, if $\partial f/\partial y \neq 0$, we obtain

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

In summary, we have the following result.

**Theorem 0.3** If the equation $f(x, y) = 0$ defines $y$ implicitly as a differentiable function of $x$, and if $\partial f/\partial y \neq 0$, then

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}. \tag{7}$$

**Example 0.3** Given that

$$x^3 + xy^2 - 3 = 0$$

find $dy/dx$ using (7), and check the result using implicit differentiation (you learned in Calculus).

**Solution** By (7) with $f(x, y) = x^3 + xy^2 - 3$,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 + y^2}{2xy}.$$

Alternatively, differentiating the given equation implicitly yields

$$3x^2 + y^2 + x\left(2y\frac{dy}{dx}\right) - 0 = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy}$$

which agress with the result obtained by (7).

Related Rates Problems (A revisit)

Theorems 0.1 and 0.2 provide us with additional perspective on related rates problems.

**Example 0.4** At what rate is the volume of a box changing if its length is 8 ft and increasing at 3 ft/s, its width is 6 ft and increasing at 2 ft/s, and its height is 4 ft and increasing at 1 ft/s?

**Solution** Let $x$, $y$, and $z$ denote the length, width, and height of the box, respectively, and let $t$ denote time in seconds. We can interpret the given rates to mean that

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 2, \quad \text{and} \quad \frac{dz}{dt} = 1 \tag{8}$$

at the instant when

$$x = 8, \quad y = 6, \quad \text{and} \quad z = 4 \tag{9}$$

We want to find $dV/dt$ at that instant. For this purpose we use the volume formula $V = xyz$ to obtain

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt} = yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt}$$

Substituting (8) and (9) into this equation yields

$$\frac{dV}{dt} = (6)(4)(3) + (8)(4)(2) + (8)(6)(1) = 184.$$

Thus, the volume is increasing at a rate of $184\,\mathrm{ft}^3/\mathrm{s}$ at the given instant.

## The Chain Rule For Partial Derivatives

In Theorem 0.1 the variables $x$ and $y$ are each functions of a single variable $t$. We now consider the case where $x$ and $y$ are each functions of two variables. Let

$$z = f(x, y) \tag{10}$$

and suppose that $x$ and $y$ are functions of $u$ and $v$, say

$$x = x(u, y), \quad y = y(u, v)$$

On substituting these functions of $u$ and $v$ in (10), we obtain the relationship

$$z = f(x(u, v), y(u, v))$$

which expresses $z$ as a function of the two variables $u$ and $v$. Thus, we can ask for the partial derivatives $\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$.

**Theorem 0.4** [Two-Variable Chain Rule]. If $x = x(u,v)$ and $y = y(u,v)$ have first-order partial derivatives at the point $(u,v)$, and if $z = f(x,y)$ is differentiable at the point $(x,y) = (x(u,v), y(u,v))$, then $z = f(x(u,v), y(u,v))$ has first-order partial derivatives at $(u,v)$ given by

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

**Example 0.5** suppose that

$$w = e^{xyz}, \quad x = 3u + v, \quad y = 3u - v, \quad z = u^2 v.$$

Use appropriate forms of the chain rule to find $\partial w/\partial u$ and $\partial w/\partial v$.

**Solution**

$$\begin{aligned}\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} = yz\,e^{xyz}(3) + xz\,e^{xyz}(3) + xy\,e^{xyz}(2uv) \\ &= e^{xyz}\,(3yz + 3xz + 2xyuv)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} = yz\,e^{xyz}(1) + xz\,e^{xyz}(-1) + xy\,e^{xyz}(u^2) \\ &= e^{xyz}\,(yz - xz + xyu^2)\end{aligned}$$

If desired, we can express $\partial w/\partial u$ and $\partial w/\partial v$ in terms of $u$ and $v$ alone by replacing $x$, $y$ and $z$ by their expression in terms of $u$ and $v$.

Other versions of the Chain Rule

Although we will not prove it, the chain rule extends to functions $w = f(u_1, u_2, \ldots, u_n)$ of $n$ variables. For example, if each $u_i$ is a function of $t$, $i = 1, 2, \ldots, n$, the relevant formula is

$$\frac{dw}{dt} = \frac{\partial w}{\partial u_1}\frac{du_1}{dt} + \frac{\partial w}{\partial u_2}\frac{du_2}{dt} + \cdots + \frac{\partial w}{\partial u_n}\frac{du_n}{dt} = \sum_{i=1}^{n} \frac{\partial w}{\partial u_i}\frac{du_i}{dt} \tag{11}$$

Note that (11) is a natural extension of Formula (3) in Theorem 0.1 and Formula (4) in Theorem 1.2.

There are infinitely many variation of the chain rule depending on the number of variables and the choice of independent and dependent variables. A good working procedure is to use tree diagrams (shown in class) to derive new versions of the chain rule as needed. This approach will give correct results for the function that we will usually encounter.

### The General Chain Rule

If $w = f(u_1, u_2, \ldots, u_n)$ is a function of $n$ variables, and each $u_i$ is a function of $m$ variables, say $t_1, t_2, \ldots, t_m$, the relevant formulas is: For each variable $t_j$ where $j = 1, 2, \ldots, m$ we have the $m$ equations below

$$\begin{aligned}
\frac{\partial w}{\partial t_1} &= \frac{\partial w}{\partial u_1}\frac{\partial u_1}{\partial t_1} + \frac{\partial w}{\partial u_2}\frac{\partial u_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial u_n}\frac{\partial u_n}{\partial t_1} = \sum_{i=1}^{n} \frac{\partial w}{\partial u_i}\frac{\partial u_i}{\partial t_1} \\
\frac{\partial w}{\partial t_2} &= \frac{\partial w}{\partial u_1}\frac{\partial u_1}{\partial t_2} + \frac{\partial w}{\partial u_2}\frac{\partial u_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial u_n}\frac{\partial u_n}{\partial t_2} = \sum_{i=1}^{n} \frac{\partial w}{\partial u_i}\frac{\partial u_i}{\partial t_2} \\
&\vdots \\
\frac{\partial w}{\partial t_j} &= \frac{\partial w}{\partial u_1}\frac{\partial u_1}{\partial t_j} + \frac{\partial w}{\partial u_2}\frac{\partial u_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial u_n}\frac{\partial u_n}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial w}{\partial u_i}\frac{\partial u_i}{\partial t_j} \\
&\vdots \\
\frac{\partial w}{\partial t_m} &= \frac{\partial w}{\partial u_1}\frac{\partial u_1}{\partial t_m} + \frac{\partial w}{\partial u_2}\frac{\partial u_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial u_n}\frac{\partial u_n}{\partial t_m} = \sum_{i=1}^{n} \frac{\partial w}{\partial u_i}\frac{\partial u_i}{\partial t_m}
\end{aligned}$$

**Example 0.6** Suppose that $w = x^2 + y^2 - z^2$ and

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi \quad \text{(Spherical coordinates)}$$

Use appropriate forms of the chain rule to find $\partial w/\partial\rho$, $\partial w/\partial\theta$ only.

**Solution**

$$\begin{aligned}
\frac{\partial w}{\partial \rho} &= 2x \sin\phi \cos\theta + 2y \sin\phi \sin\theta - 2z \cos\phi \\
&= 2\rho \sin^2\phi \cos^2\theta + 2\rho \sin^2\phi \sin^2\theta - 2\rho \cos^2\phi \\
&= 2\rho \sin^2\phi (\cos^2\theta + \sin^2\theta) - 2\rho \cos^2\phi \\
&= 2\rho (\sin^2\phi - \cos^2\phi) \\
&= -2\rho \cos 2\phi
\end{aligned}$$

$$\begin{aligned}
\frac{\partial w}{\partial \theta} &= -2x\,\rho \sin\phi \sin\theta + 2y\,\rho \sin\phi \cos\theta - 2z\,\rho(0) \\
&= -2\rho^2 \sin^2\phi \sin\theta \cos\theta + 2\rho^2 \sin^2\phi \sin\theta \cos\theta \\
&= 0
\end{aligned}$$

This result is explained by the fact that $w$ does not vary with $\theta$. You can see this directly by expressing the variables $x$, $y$, and $z$ in term of $\rho$, $\phi$, and $\theta$ in the formula for $w$. (Verify that $w = -2\rho^2 \cos 2\phi$.)

**Example 0.7** Suppose that

$$w = xy + yz, \quad y = \sin x, \quad z = e^x$$

Use an appropriate form of the chain rule to find $dw/dx$.

**Solution**

$$\begin{aligned}
\frac{dw}{dx} &= y + (x + z) \cos x + y\,e^x \\
&= \sin x + (x + e^x) \cos x + e^x \sin x \\
&= x \cos x + \sin x + e^x (\cos x + \sin x)
\end{aligned}$$

This result can also be obtained by first expressing $w$ explicitly in terms of $x$ as

$$w = x \sin x + e^x \sin x$$

and then differentiating with respect to $x$; however, such direct substitutions is not always possible.

In each of the expressions

$$z = \sin xy, \quad z = \frac{xy}{1 + xy}, \quad z = e^{xy}$$

the independent variables occur only in the combination $xy$, so the substitution $t = xy$ reduces the expression to a function of a single variable:

$$z = \sin t, \quad z = \frac{t}{1 + t}, \quad z = e^{t}$$

Conversely, if we begin with a function of one variable $z = f(t)$ and substitute $t = xy$, we obtain a function $z = f(xy)$ in which the variables appear only in the combination $xy$. Functions whose variables occur in fixed combinations arise frequently in applications.

**Example 0.8** Show that when $f$ is differentiable, a function of the form $z = f(xy)$ satisfies the equation

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$$

**Solution** Let $t = xy$, so that $z = f(t)$. From the chain rule we obtain

$$\frac{\partial z}{\partial x} = \frac{dz}{dt}\frac{\partial t}{\partial x} = y\frac{dz}{dt} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{dz}{dt}\frac{\partial t}{\partial y} = x\frac{dz}{dt}$$

from which it follows that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = xy\frac{dz}{dt} - yx\frac{dz}{dt} = 0$$

Section 1

# Directional Derivatives and Gradients

The partial derivatives $f_x(x,y)$ and $f_y(x,y)$ represent the rates of change of $f(x,y)$ in directions parallel to the $x$- and $y$-axes. In this discussion we will investigate rates of change of $f(x,y)$ in other directions.

What do we mean by a Directional Derivative of a function $z = f(x,y)$?

In this discussion we extend the concept of a *partial* derivative to the more general notion of a *directional* derivative. We have seen (in the classroom) that partial derivatives of a function give the instantaneous rates of change of that function in directions parallel to the coordinate axes. Directional derivatives allow us to compute the rates of change of a function with respect to distance in *any* direction.

The directional derivative problem: Suppose that we wish to compute the instantaneous rate of change of a function $z = f(x,y)$ with respect to the distance from a point $(x_0, y_0)$ in some direction. Since there are infinitely many different directions from the point $(x_0, y_0)$ in which we could move, we need a convenient method for describing a specific direction starting at $(x_0, y_0)$. One way to do this is to use a **unit vector**

$$\hat{\mathbf{u}} = u_1\,\hat{\mathbf{i}} + u_2\,\hat{\mathbf{j}}$$

that has its "tail" or initial point at $(x_0, y_0)$ and points in the desired direction.

This unit vector $\hat{\mathbf{u}} = u_1\,\hat{\mathbf{i}} + u_2\,\hat{\mathbf{j}}$ determines a line $l$ (of action) passing through $(x_0, y_0)$ in the $xy$-plane, that can be expressed with a position vector function

$$\vec{\mathbf{r}} = (x, y)$$

with the component functions $x$ and $y$ are given parametrically as

$$x = x_0 + s\,u_1, \quad y = y_0 + s\,u_2 \tag{1}$$

where $s$ is the arc-length parameter (recall that the arc-length is the distance measured along a curve) that has its reference point at $\vec{\mathbf{r}}_0 = (x_0, y_0)$ and has positive values in the direction of $\hat{\mathbf{u}}$. For $s = 0$, the point $\vec{\mathbf{r}} = (x, y)$ is at the reference point $\vec{\mathbf{r}}_0 = (x_0, y_0)$, and as $s$ increases, the point $(x, y)$ moves along the line $l$ in the direction of $\hat{\mathbf{u}}$. On the line $l$ the variable $z = f(\vec{\mathbf{r}}) = f(x_0 + s\,u_1, y_0 + s\,u_2)$ is a function of the parameter $s$. The value of the derivative $dz/ds$ at $s = 0$ then gives an instantaneous rate of change of $f(x,y)$ with respect to distance $s$ along the line $l$ from the point $(x_0, y_0)$ in the direction of $\hat{\mathbf{u}}$.

**Definition 1.1** [DIRECTIONAL DERIVATIVE] If $z = f(x, y)$ is a function of $x$ and $y$, and if $\mathbf{\hat{u}} = u_1\,\mathbf{\hat{i}} + u_2\,\mathbf{\hat{j}}$ is a unit vector, then the **directional derivative of $f$ in the direction of $\mathbf{\hat{u}}$** at $(x_0, y_0)$ is denoted by

$$D_{\mathbf{\hat{u}}} f(x_0, y_0) = \frac{d}{ds}[f(x_0 + s\,u_1, y_0 + s\,u_2)]_{s=0} \tag{2}$$

provided this derivative exists.

GEOMETRIC INTERPRETATION OF THE DIRECTIONAL DERIVATIVE: Geometrically, $D_{\mathbf{\hat{u}}} f(x_0, y_0)$ can be interpreted as the **slope of the surface $z = f(x, y)$ in the direction of $\mathbf{\hat{u}}$** at the point $(x_0, y_0, f(x_0, y_0))$. Usually the value of $D_{\mathbf{\hat{u}}} f(x_0, y_0)$ will depend on both the point $(x_0, y_0)$ and the direction $\mathbf{\hat{u}}$. Thus, at a fixed point the slope of the surface may vary with the direction. Analytically, the directional derivative represents the **instantaneous rate of change of $z = f(x, y)$ with respect to distance in the direction of $\hat{u}$** at the point $(x_0, y_0)$

**Example 1.1** Let $f(x, y) = xy$ and find $D_{\mathbf{\hat{u}}} f(1, 2)$, where $\mathbf{\hat{u}} = \frac{\sqrt{3}}{2}\,\mathbf{\hat{i}} + \frac{1}{2}\,\mathbf{\hat{j}}$.

**Solution** It follows from Equation (2) above that

$$D_{\mathbf{\hat{u}}} f(1, 2) = \frac{d}{ds}\left[f\left(1 + \frac{\sqrt{3}}{2}\,s, 2 + \frac{1}{2}\,s\right)\right]_{s=0}$$

Since

$$f\left(1 + \frac{\sqrt{3}}{2}\,s, 2 + \frac{1}{2}\,s\right) = \left(1 + \frac{\sqrt{3}}{2}\,s\right)\left(2 + \frac{1}{2}\,s\right) = 2 + \left(\sqrt{3} + \frac{1}{2}\right)s + \frac{\sqrt{3}}{4}\,s^2$$

we have

$$\frac{d}{ds}\left[f\left(1 + \frac{\sqrt{3}}{2}\,s, 2 + \frac{1}{2}\,s\right)\right] = \frac{\sqrt{3}}{2}\,s + \frac{1}{2} + \sqrt{3}$$

and thus

$$\frac{d}{ds}\left[f\left(1 + \frac{\sqrt{3}}{2}\,s, 2 + \frac{1}{2}\,s\right)\right]_{s=0} = \frac{1}{2} + \sqrt{3}$$

Since $\frac{1}{2} + \sqrt{3} \approx 2.23$, we conclude that if we move a small distance from the point $(1, 2)$ in the direction of $\mathbf{\hat{u}}$, the function $f(x, y) = xy$ will increase by about 2.23 times the distance moved.

The definition of a directional derivative for a function $f(x, y, z)$ of three variables is similar to Definition 1.1

**Definition 1.2** If $\hat{\mathbf{u}} = u_1\,\hat{\mathbf{i}} + u_2\,\hat{\mathbf{j}} + u_3\,\hat{\mathbf{k}}$ is a unit vector, and if $w = f(x,y,z)$ is a function of $x$,$y$, and $z$, then the **directional derivative of $f$ in the direction of $\hat{\mathbf{u}}$** at $(x_0, y_0, z_0)$ is denoted by

$$D_{\hat{\mathbf{u}}} f(x_0, y_0, z_0) = \frac{d}{ds}[f(x_0 + s\,u_1, y_0 + s\,u_2, z_0 + s\,u_3)]_{s=0} \tag{3}$$

provided this derivative exists.

Although Equation (3) does not have a convenient geometric interpretation, we can still interpret directional derivatives for functions of three variables in terms of instantaneous rates of change in a specified direction.

For a function that is differentiable at a point, directional derivatives exists in every direction from the point and can be computed directly in terms of the first-order partial derivatives of the function.

## Theorem 1.1

(a) If $f(x,y)$ is differentiable at $(x_0, y_0)$ and if $\hat{\mathbf{u}} = u_1\,\mathbf{i} + u_2\,\hat{\mathbf{j}}$ is a unit vector, then the directional derivative $D_{\hat{\mathbf{u}}} f(x_0, y_0)$ exists and is given by

$$D_{\hat{\mathbf{u}}} f(x_0, y_0) = f_x(x_0, y_0)\,u_1 + f_y(x_0, y_0)\,u_2 \tag{4}$$

(b) If $f(x,y,z)$ is differentiable at $(x_0, y_0, z_0)$ and if $\hat{\mathbf{u}} = u_1\,\mathbf{i} + u_2\,\hat{\mathbf{j}} + u_3\,\hat{\mathbf{k}}$ is a unit vector, then the directional derivative $D_{\hat{\mathbf{u}}} f(x_0, y_0, z_0)$ exists and is given by

$$D_{\hat{\mathbf{u}}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)\,u_1 + f_y(x_0, y_0, z_0)\,u_2 + f_z(x_0, y_0, z_0)\,u_3 \tag{5}$$

We can use this theorem to confirm the result of Example 1.1. For $f(x,y) = xy$ we have $f_x(1,2) = 2$ and $f_y(1,2) = 1$. With $\hat{\mathbf{u}} = \frac{\sqrt{3}}{2}\,\hat{\mathbf{i}} + \frac{1}{2}\,\hat{\mathbf{j}}$. Equation (4) becomes

$$D_{\hat{\mathbf{u}}} f(1,2) = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2} = \sqrt{3} + \frac{1}{2}$$

which agrees with our solution in Example 1.1.

Recall that a unit vector $\hat{\mathbf{u}} = u_1\,\hat{\mathbf{i}} + u_2\,\hat{\mathbf{j}}$ in the $xy$-plane can be expressed as

$$\hat{\mathbf{u}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}} \tag{6}$$

where $\phi$ is the angle from the positive $x$-axis to $\hat{\mathbf{u}}$. Thus, Formula (4) can also be expressed as

$$D_{\hat{\mathbf{u}}} f(x_0, y_0) = f_x(x_0, y_0)\,\cos\phi + f_y(x_0, y_0)\,\sin\phi \tag{7}$$

**Example 1.2** Find the directional derivative of $f(x, y) = e^{xy}$ at $(-2, 0)$ in the direction of the unit vector that makes an angle of $\pi/3$ with the positive $x$-axis.

**Solution** The partial derivatives of $f$ are

$$f_x(x, y) = y\,e^{xy}, \qquad f_y(x, y) = x\,e^{xy}$$

$$f_x(-2, 0) = 0, \qquad f_y(-2, 0) = -2$$

The unit vector $\hat{\mathbf{u}}$ that makes an angle of $\pi/3$ with the positive $x$-axis is

$$\hat{\mathbf{u}} = \cos(\pi/3)\,\hat{\mathbf{i}} + \sin(\pi/3)\,\hat{\mathbf{j}} = \frac{1}{2}\,\mathbf{i} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}$$

Thus, from (7)

$$\begin{aligned} D_{\hat{\mathbf{u}}}f(-2, 0) &= f_x(-2, 0)\,\cos(\pi/3) + f_y(-2, 0)\,\sin(\pi/3) \\ &= 0(1/2) + (-2)(\sqrt{3}/2) = -\sqrt{3} \end{aligned}$$

It is important that the directional derivative be specified by a unit vector when applying either Equation (4) or Equation (5).

**Example 1.3** Find the directional derivative of $f(x, y, z) = x^2y - yz^3 + z$ at the point $(1, -2, 0)$ in the direction of $\vec{\mathbf{a}} = 2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}$.

**Solution** The partial derivatives of $f$ are

$$f_x(x, y, z) = 2xy, \qquad f_y(x, y) = x^2 - z^3, \qquad f_z(x, y, z) = 3yz^2 + 1$$

$$f_x(1, -2, 0) = -4, \qquad f_y(1, -2, 0) = 1, \qquad f_z(1, -2, 0) = 1$$

Since $\vec{\mathbf{a}}$ is not a unit vector, we normalize it, getting

$$\hat{\mathbf{u}} = \frac{\vec{\mathbf{a}}}{\|\,\vec{\mathbf{a}}\,\|} = \frac{1}{\sqrt{9}}\,(2\,\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\,\hat{\mathbf{k}}) = \frac{2}{3}\,\hat{\mathbf{i}} + \frac{1}{3}\,\hat{\mathbf{j}} - \frac{2}{3}\,\hat{\mathbf{k}}$$

Formula (5) then yields

$$D_{b\hat{f}u}f(1, -2, 0) = (-4)\left(\frac{2}{3}\right) + \frac{1}{3} - \frac{2}{3} = -3$$

The Gradient

Formula (4) can be expressed in the form of a dot product as

$$\begin{aligned} D_{\hat{\mathbf{u}}} f(x_0, y_0) &= (f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}}) \cdot (u_1\,\hat{\mathbf{i}} + u_2\,\hat{\mathbf{j}}) \\ &= (f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}}) \cdot \hat{\mathbf{u}} \end{aligned}$$

Similarly, Formula (5) can be expressed as

$$D_{\hat{\mathbf{u}}} f(x_0, y_0, z_0) = (f_x(x_0, y_0, z_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0, z_0)\,\hat{\mathbf{j}} + f_z(x_0, y_0, z_0)\,\hat{\mathbf{k}}) \cdot \hat{\mathbf{u}}$$

In both cases the directional derivative is obtained by dotting the direction vector $\hat{\mathbf{u}}$ with a new vector constructed from the first-order partial derivatives of $f$.

## Definition 1.3 [The gradient of a function]

(a) If $f$ is a function of $x$ and $y$, then the **gradient of** $f$ is defined by

$$\nabla f(x, y) = f_x(x, y)\,\hat{\mathbf{i}} + f_y(x, y)\,\hat{\mathbf{j}} \tag{8}$$

(b) if $f$ is a function of $x$, $y$, and $z$, then the **gradient of** $f$ is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}} \tag{9}$$

The symbol $\nabla$ (read "del") is an inverted delta. (It is sometimes called a "nabla" because of its similarity in form to an ancient Hebrew ten-stringed harp of that name.)

Formulas (4) and (5) can now be written as

$$D_{\hat{\mathbf{u}}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{\mathbf{u}} \tag{10}$$

and

$$D_{\hat{\mathbf{u}}} f(x_0, y_0, z_0) = \nabla f(x_0, y_0, z_0) \cdot \hat{\mathbf{u}} \tag{11}$$

respectively. For example, using Formula (11) our solution to Example 1.3 would take the form

$$D_{\hat{\mathbf{u}}} f(1, -2, 0) = \nabla f(1, -2, 0) \cdot \hat{\mathbf{u}} = (-4\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot \left(\frac{2}{3}\,\hat{\mathbf{i}} + \frac{1}{3}\,\hat{\mathbf{j}} - \frac{2}{3}\,\hat{\mathbf{k}}\right) = -3$$

Formula (10) can be interpreted to mean that the slope of the surface $z = f(x, y)$ at the point $(x_0, y_0)$ in the direction of $\hat{\mathbf{u}}$ is the dot product of the gradient with $\hat{\mathbf{u}}$.

### Properties of the Gradient

The gradient is not merely a notational device to simplify the formula for the directional derivative, we will see that the length and direction of the gradient $\nabla f$ provide important information about the function $f$ and the surface $z = f(x, y)$. For example, suppose that $\nabla f(x, y) \neq \mathbf{0}$, and let us use our known formulas for the dot product to rewrite Equation (10) as

$$D_{\hat{\mathbf{u}}} f(x, y) = \nabla f(x, y) \cdot \hat{\mathbf{u}} = \| \nabla f(x, y) \| \, \| \hat{\mathbf{u}} \| \cos\theta = \| \nabla f(x, y) \| \cos\theta \tag{12}$$

where $\theta$ is the angle between $\nabla f(x, y)$ and $\hat{\mathbf{u}}$. This equation tells us that the maximum value of $D_{\hat{\mathbf{u}}} f(x, y)$ is $\| \nabla f(x, y) \|$, and this maximum occurs when $\theta = 0$, that is, when $\hat{\mathbf{u}}$ is in the direction of $\nabla f(x, y)$. Geometrically, this means that the *surface $z = f(x, y)$ has its maximum slope at a point $(x, y)$ in the direction of the gradient, and the maximum slope is $\| \nabla f(x, y) \|$*. Similarly, (12) tells us that the minimum value of $D_{\hat{\mathbf{u}}} f(x, y)$ is $-\| \nabla f(x, y) \|$, and this minimum occurs when $\theta = \pi$, that is, when $\hat{\mathbf{u}}$ is oppositely directed to $\nabla f(x, y)$. Geometrically, this means that the *surface $z = f(x, y)$ has its minimum slope at a point $(x, y)$ in the opposite direction of the gradient, and the minimum slope is $- \| \nabla f(x, y) \|$*.

Finally, in the case where $\nabla f(x, y) = \mathbf{0}$, it follows from (12) that $D_{\hat{\mathbf{u}}} f(x, y) = 0$ in all directions at the point $(x, y)$. This typically occurs where the surface $z = f(x, y)$ has a "relative maximum," a "relative minimum," or a saddle point.

A similar analysis applies to functions of three variables. As a consequence, we have the following result.

**Theorem 1.2** . Let $f$ be a function of either two variables or three variables. and let $P$ denote the point $P(x_0, y_0)$ or $P(x_0, y_0, z_0)$, respectively. Assume that $f$ is differentiable at $P$.

a) If $\nabla f = \mathbf{0}$ at P, then all directional derivatives of $f$ at $P$ are zero.

b) If $\nabla f \neq \mathbf{0}$ at $P$, then among all possible directional derivatives of $f$ at $P$, the derivative in the direction of $\nabla f$ has the largest value. The value of this largest directional derivative is $\| \nabla f \|$ at $P$.

c) If $\nabla f \neq \mathbf{0}$ at $P$, then among all possible directional derivatives of $f$ at $P$, the derivative in the direction opposite that of $\nabla f$ has the smallest value. The value of this smallest directional derivative is $-\| \nabla f \|$ at $P$.

**Example 1.4** Let $f(x,y) = x^2\, e^y$. Find the maximum value of a directional derivative at $(-2,0)$, and find the unit vector in the direction in which the maximum value occurs.

**Solution** Since

$$\nabla f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} = 2xe^y\,\hat{\mathbf{i}} + x^2e^y\,\hat{\mathbf{j}}$$

the gradient of $f$ at $(-2,0)$ is

$$\nabla f(-2,0) = -4\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}.$$

By Theorem 1.2, the maximum value of the directional derivative is

$$\|\,\nabla f(-2,0)\,\| = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$$

This maximum value occurs in the direction of $\nabla f(-2,0)$. The unit vector in this direction is

$$\hat{\mathbf{u}} = \frac{\nabla f(-2,0)}{\|\,\nabla f(-2,0)\,\|} = \frac{1}{4\sqrt{2}}(-4\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}) = -\frac{1}{\sqrt{2}}\,\hat{\mathbf{i}} + \frac{1}{\sqrt{2}}\,\hat{\mathbf{j}}$$

## Gradients are Normal to Level Curves

We have seen that *the gradient vector of a function points in the direction in which a function increases most rapidly.*

For a function $f(x,y)$ of two variables, we will now consider how this direction of maximum rate of increase can be determined from a contour map of the function.

Suppose that $(x_0, y_0)$ is a point on a level curve of $f$ given by the equation $f(x,y) = c$. Assume that such a level curve can be smoothly parametrized as

$$x = x(s), \quad y = y(s) \tag{13}$$

where $s$ is an arc length parameter (the distance along the level curve). Recall from earlier discussions that the unit tangent vector to (13) is

$$\vec{\mathbf{T}} = \vec{\mathbf{T}}(s) = \frac{dx}{ds}\,\hat{\mathbf{i}} + \frac{dy}{ds}\,\hat{\mathbf{j}}$$

Since $\vec{\mathbf{T}}$ gives a direction along which $f$ is nearly constant, we would expect the instantaneous rate of change of $f$ with respect to distance in the direction of $\vec{\mathbf{T}}$ to be 0. That is, we would expect that

$$D_{\vec{\mathbf{T}}} f(x,y) = \nabla f(x,y)\cdot\vec{\mathbf{T}}(s) = 0.$$

To show this to be the case, we differentiate both sides of the equation $f(x, y) = c$ with respect to $s$. Assuming that $f$ is differentiable at $(x, y)$, we can use the chain rule to obtain

$$\frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = 0$$

which wee can rewrite as

$$\left(\frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}}\right) \cdot \left(\frac{dx}{ds}\,\hat{\mathbf{i}} + \frac{dy}{ds}\,\hat{\mathbf{j}}\right) = 0$$

or, alternatively as

$$\nabla f(x, y) \cdot \vec{\mathbf{T}} = 0.$$

Therefore, if $\nabla f(x, y) \neq \mathbf{0}$, then $\nabla f(x, y)$ should be normal to the level curve $f(x, y) = c$ at any point $(x, y)$ on the curve.

It has been proved in advance calculus courses that if $f(x, y)$ has continuous first-order partial derivatives at $(x_0, y_0)$, and if this first-order partial derivatives are not simultaneously 0 near $(x_0, y_0)$, that is $\nabla f(x_0, y_0) \neq \mathbf{0}$ near $(x_0, y_0)$ the graph of $f(x, y) = c$ (a level curve of $f$) is indeed a smooth curve passing through $(x_0, y_0)$. Furthermore, we also know from our other discussions that $f$ will be differentiable at $(x_0, y_0)$ (Recall that differentiability means that the first-order partial derivatives exists and are continuous at $(x_0, y_0)$). We therefore have the following result:

**Theorem 1.3** Let $f(x, y)$ be a function with continuous first-order partial derivatives in an open disk centered at $(x_0, y_0)$ and that $\nabla f(x_0, y_0) \neq \mathbf{0}$. Then $\nabla f(x_0, y_0)$ is normal to the level curve of $f$ which passes through $(x_0, y_0)$.

When we examine a contour map, we instinctively regard the distance between adjacent contours to be measured in a normal direction. If the contours correspond to equally spaced values of $f$, then the closer together the contours apear to be, the more rapidly the values of $f$ will be changing in that normal direction. It follows from Theorems 1.2 and 1.3 that this rate of change of $f$ is given by $\| \nabla f(x, y) \|$. Thus the closer together the contours appear to be, the greater the length of the gradient vector $\nabla f$.

*If $(x_0, y_0)$ is a point on the level curve $f(x, y) = c$, then the slope of the surface $z = f(x, y)$ at that point in the direction of a unit vector $\hat{\mathbf{u}}$ is*

$$D_{\hat{\mathbf{u}}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \hat{\mathbf{u}}.$$

*If $\hat{\mathbf{u}}$ is tangent to the level curve at $(x_0, y_0)$, then $f(x, y)$ is neither increasing nor decreasing in that direction, so $D_{\hat{\mathbf{u}}} f(x_0, y_0) = 0$. Thus, $\nabla f(x_0, y_0)$, $-\nabla f(x_0, y_0)$, and the tangent vector $\hat{\mathbf{u}}$ mark the directions of maximum slope, minimum slope, and zero slope at a point $(x_0, y_0)$ on a level curve. Good skiers use these facts intuitively to control their speed by zigzagging down ski slopes–the ski across the slope with their skis tangential to a level curve to stop their downhill motion, and they point their skis down the slope and normal to the level to obtain the most rapid descent.*

An Application of Gradients

There are numerous applications in which the motion of an object must be controlled so that it moves towards a heat source. For example, in medical applications the operation of certian diagnostic equipment is designed to locate heat sources generated by tumors or infections, and in military applications the trajectories of heat-seeking missiles are controlled to seek and destroy enemy aircraft. The following example illustrates how gradients are used to solve such problems.

**Example 1.5** A heat-seeking particle is located at the point $(2, 3)$ on a flat metal plate whose temperature at a point $(x, y)$ on the plate is

$$T(x, y) = 10 - 8x^2 - 2y^2$$

Find an equation for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

**Solution** Let us assume that the trajectory is represented parametrically by the equations

$$x = x(t), \quad y = y(t)$$

where the particle is at the point $(x(0), y(0)) = (2, 3)$ at time $t = 0$. Because the particle moves in the direction of maximum temperature increase, its direction of motion at time $t$ is in the direction of the gradient of $T(x, y)$, and hence its velocity vector $\mathbf{v}(t)$ at time $t$ points in the direction of the gradient. Thus, there is a scalar $k$ that depends on $t$ such that

$$\mathbf{v}(t) = k\,\nabla T(x, y) = k\,(-16x\,\hat{\mathbf{i}} - 4y\,\hat{\mathbf{j}})$$

from which we obtain

$$\frac{dx}{dt}\,\hat{\mathbf{i}} + \frac{dy}{dt}\,\hat{\mathbf{j}} = k\,(-16x\,\hat{\mathbf{i}} - 4y\,\hat{\mathbf{j}})$$

Equating components yields

$$\frac{dx}{dt} = -16kx, \quad \frac{dy}{dt} = -4ky$$

and dividing to eliminate $k$ yields

$$\frac{dy}{dx} = \frac{-4ky}{-16kx} = \frac{y}{4x}.$$

Thus, we can obtain the trajectory by solving the initial-value problem

$$\frac{dy}{dx} - \frac{y}{4x} = 0, \quad y(2) = 3$$

The differential equation is a separable first-order linear equation and hence can be solve by separating the variables (this you know alot about, this was what calculus II was about) or by the method of integrating factors (this you know nothing about yet!)

$$\frac{dy}{y} - \frac{dx}{4x} = d(lnC), \quad y(2) = 3$$

Now by integration we obtain that

$$\ln y - \ln \sqrt[4]{x} = \ln\left(\frac{y}{\sqrt[4]{x}}\right) = \ln C$$

From which it follows that

$$y = C\sqrt[4]{x}$$

. Thus if $x = 2$, then $y = C\sqrt[4]{2} = 3$ implying that $C = \frac{3}{\sqrt[4]{2}}$ and so the equation of the trajectory is

$$y = \frac{3}{\sqrt[4]{2}}\sqrt[4]{x}$$

Section 2

# Problem #1

---

A **homogeneous function of degree n** is a function $f(x,y)$ which satisfies the equation

$$f(tx,ty) = t^n f(x,y); \quad \text{for all } t > 0.$$

For example, the function $f(x,y) = 3x^2 + y^2$ is homogeneous of degree 2 because $f(tx,ty) = 3(tx)^2 + (ty)^2 = 3t^2x^2 + t^2y^2 = t^2(3x^2 + y^2) = t^2 f(x,y)$.

a) Show that if $f(x,y)$ is a homogeneous function of degree $n$, then

$$x\,\frac{\partial f}{\partial x}(x,y) + y\,\frac{\partial f}{\partial y}(x,y) = n\,f(x,y)$$

[*Hint*: partially differentiate both sides of $f(tx,ty) = t^n f(x,y)$ with respect to $t$.]

b) Show that if $f(x,y)$ is homogeneous of degree $n$, then

$$x^2\,\frac{\partial^2 f}{\partial x^2}(x,y) + 2xy\,\frac{\partial^2 f}{\partial x\,\partial y}(x,y) + y^2\,\frac{\partial^2 f}{\partial x^2}(x,y) = n(n-1)\,f(x,y)$$

c) Show that if $f(x,y)$ is homogeneous of degreen $n$, then its first-order partial derivative with respect $x$, $f_x(x,y)$ is homogeneous of degree $n-1$ (i.e., $f_x(tx,ty) = t^{n-1}\,f_x(x,y)$)

Section 3
## Problem #2

---

a) Let $w = 3xy^2z^3$, $y = 3x^2 + 2$, $z = \sqrt{x - 1}$. Find $dw/dx$.

b) Suppose that the portion of a tree that is usable for lumber is a right circular cylinder. If the usable height of a tree increases 2 ft per year and the usable diameter increases 3 in per year, how fast is the volume of usable lumber increasing the instant the usable height of the tree is 20 ft and the usable diameter is 30 in?

Section 4

# Problem #3

---

Use Theorem 1.3 to find $dy/dx$ and check your result using implicit differentiation.

Section 5

## Problem #4

---

The temperature (in degrees Celsius) at a point $(x, y)$ on a metal plate in the $xy$-plane is

$$T(x, y) = \frac{xy}{1 + x^2 + y^2}$$

a) Find the rate of change of temperature at $(1, 1)$ in the direction of $\mathbf{a} = 2\,\hat{\mathbf{i}} - \hat{\mathbf{j}}$.

b) An ant at $(1, 1)$ wants to walk in the direction in which the temperature drops most rapidly. Find a unit vector in that direction.

Section 6

# Problem #5

---

If the electric potential at a point $(x, y)$ in the $xy$-plane is $V(x, y)$, then the **electric intensity vector** at the point $(x, y)$ is $\mathbf{E} = -\nabla V(x, y)$. Suppose that $V(x, y) = e^{-2x} \cos 2y$.

a) Find the electric intensity vector at $(\pi/4, 0)$.

b) Show that at each point in the plane, the electric potential decreases most rapidly in the direction of the vector $\mathbf{E}$.

SECTION 7

## Problem #6

---

Prove and commit to your memory: If $f$ and $g$ are differentiable, then

a) $\nabla(f + g) = \nabla f + \nabla g$.

b) $\nabla(\alpha f) = \alpha \nabla f$. ($\alpha$ is a constant)

c) $\nabla(f\,g) = f\,\nabla g + g\,\nabla f$.

d ) $\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2}$