Introduction

In this discussion we will extend many of the basic concepts of calculus to functions of two or more variables, commonly called functions of several variables. We will begin by discussing limits and continuity for functions of two and three variables, then we will define derivatives of such functions, and then we will use these derivatives to study differentiability, continuity and local linearity problems. Although many of the basic ideas that we developed for functions of one variable will carry over in a natural way, functions of several variables are intrinsically more complicated than functions of one variable, so we will need to develop new tools and new ideas to deal with such functions.

Functions of Two or More Variables

In calculus and in previous discussion, we studied real-valued functions of a real variable and vector-valued functions of a real variable. In this discussion we will consider real-valued functions of two or more variables.

Notation and Terminology

There are many familiar formulas in which a given variable depends on two or more other variables. For example, the area $A$ of a triangle depends on the base length $b$ and height $h$ by the formula $A = \frac{1}{2} bh$; the volume $V$ of a rectangular box depends on the length $l$, the width $w$, and the height $h$ by the formula $V = lwh$; and the arithmetic average $\bar{x}$ of $n$ real numbers, $x_1, x_2, \ldots, x_n$, depends on those numbers by the formula

$$\bar{x} = \frac{1}{n} (x_1 + x_2 + \cdots + x_n).$$

Thus, we say that

- $A$ is a function of the two variables $b$ and $h$;
- $V$ is a function of the three variables $l$, $w$, and $h$;
- $\bar{x}$ is a function of the $n$ variables $x_1, x_2, \ldots, x_n$.

The terminology and notation for functions of two or more variables is similar to that for functions of one variable. For example, the expression

$$z = f(x, y)$$

means that $z$ is a function of $x$ and $y$ in the sense that a unique value of the dependent variable $z$ is determined by specifying values for the independent variables $x$ and $y$. Similarly,

$$w = f(x, y, z)$$

expresses $w$ as a function of $x$, $y$, and $z$, and

$$u = f(x_1, x_2, \ldots, x_n)$$
expresses $u$ as a function of $x_1, x_2, \ldots, x_n$.

We will find it useful to think of functions of two or three independent variables in geometric terms. For example, if $z = f(x, y)$, then we can view $(x, y)$ as a point in the $xy$-plane and think of $f$ as a rule that associates a unique numerical value $z$ with the point $(x, y)$; similarly, we can think of $w = f(x, y, z)$ as a rule that associates a unique numerical value $w$ with a point $(x, y, z)$ in an $xyz$-coordinate system.

As with functions of a single variable, the independent variables of a function of two or more variables may be restricted to lie in some set $D$, which we call the domain of $f$. Sometimes the domain will be determined by physical restrictions on the variables. If the function is defined by a formula and if there are no physical restrictions or other restrictions stated explicitly, then it is understood that the domain consists of all points for which the function yields a real value for the dependent variable. We call this the natural domain of the function. The following definitions summarize this discussion.

**Definition** A function $f$ of two variables, $x$ and $y$, is a rule that assigns a unique real number $f(x, y)$ to each point $(x, y)$ in some set $D$ in the $xy$-plane.

**Definition** A function $f$ of three variables, $x$, $y$, and $z$ is a rule that assigns a unique real number $f(x, y, z)$ to each point $(x, y, z)$ in some set $D$ in the 3-dimensional space.
Example -1.1 Let 

\[ f(x, y) = 3x^2 \sqrt{y} - 1 \]

Find \( f(1, 4), f(0, 9), f(t^2, t), f(ab, 9b), \) and the natural domain of \( f \).

**Solution** By substitution

\[
\begin{align*}
    f(1, 4) &= 3(1)^2 \sqrt{4} - 1 = 5 \\
    f(0, 9) &= 3(0)^2 \sqrt{9} - 1 = -1 \\
    f(t^2, t) &= 3(t^2)^2 \sqrt{t} - 1 = 3t^4 \sqrt{t} - 1 \\
    f(ab, 9b) &= 3(ab)^2 \sqrt{9b} - 1 = 9a^2 b^2 \sqrt{b} - 1.
\end{align*}
\]

Because of the radical \( \sqrt{y} \) appearing in the formula for \( f \), we must have \( y \geq 0 \) to avoid non-real values for \( f(x, y) \). Thus, the natural domain of \( f \) consists of all points in the \( xy \)-plane that are on or above the \( x \)-axis (called a half-plane).

Example -1.2 Find the natural domain of the function \( f(x, y) = \ln(x^2 - y) \).

**Solution** \( \ln(x^2 - y) \) is defined only when \( 0 < x^2 - y \) or \( y < x^2 \). We could first sketch the parabola \( y = x^2 \) as a “dashed” curve. The natural domain region \( y < x^2 \) would then consists of all points below this curve.

Example -1.3 Let 

\[ f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}. \]

Find \( f \left(0, \frac{1}{2}, -\frac{1}{2}\right) \) and the natural domain of \( f \).

**Solution** By Substitution,

\[
\begin{align*}
    f \left(0, \frac{1}{2}, -\frac{1}{2}\right) &= \sqrt{1 - (0)^2 - \left(\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}}
\end{align*}
\]

Because of the square root sign, we must have \( 0 \leq 1 - x^2 - y^2 - z^2 \) in order to have a real value for \( f(x, y, z) \). Rewriting this inequality in the form

\[ x^2 + y^2 + z^2 \leq 1 \]

we see that the natural domain of \( f \) consists of all points on or within the sphere

\[ x^2 + y^2 + z^2 = 1. \]
Partial Derivatives

Graphs of Functions of two Variables

Recall that for a function $f$ of a single variable, the graph of $f(x)$ in the $xy$-plane was defined to be the graph of the equation $y = f(x)$. Similarly, if $f$ is a function of two variables, we define the graph of $f(x, y)$ in $xyz$-space to be the graph of the equation $z = f(x, y)$. In general, such a graph will be a surface in 3-space.

Example -1.4  In each part, describe the graph of the functions in an $xyz$-coordinate system.

a) $f(x, y) = 1 - x - \frac{1}{2} y$;  
b) $f(x, y) = \sqrt{1 - x^2 - y^2}$;  
c) $f(x, y) = -\sqrt{x^2 + y^2}$.

Solution

What are Level Curves?

We are all familiar with the topographic (or contour maps in which a three-dimensional landscape, such as a mountain range, is represented by two-dimensional contour lines or curves of constant elevation. Consider, for example, a small hill. A contour map of the hill is constructed by passing planes of constant elevation through the hill, projecting the resulting contours onto a flat surface, and labeling the contours with their elevations. Contour maps are also useful for studying functions of two variables. If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$ (is a curve at level $k$). The projection of this intersection onto the $xy$-plane is called the level curve of height $k$ or the level curve with constant $k$. A set of level curves for $z = f(x, y)$ is called a contour plot or contour map of $f$.

Using Matlab to some level curves of surfaces $z = f(x, y)$

Example -1.5  The graph of the function $f(x, y) = y^2 - x^2$ in the $xyz$-space is the hyperbolic paraboloid (saddle surface). The level curves have equations of the form $f(x, y) = y^2 - x^2 = k$ where $k$ is a constant in the range of $f$. For $k > 0$ these curves are hyperbolas opening along lines parallel to the $y$-axis; for $k < 0$ they are hyperbolas opening along lines parallel to the $x$-axis; and for $k = 0$ the level curve consists of the intersecting lines $y + x = 0$ and $y - x = 0$. The matlab M-file that does this is as follows:
clear all;
close all;
clf reset;
clc;
U = -5:.40:5;
V = -5:.40:5;
s = -5:.01:5;
t = -5:.01:5;
levels = -20:5:20;
[u,v] = meshgrid(U,V);
[xx,yy]=meshgrid(s,t);
f = @(x,y)y.∧2 - x.∧2;
% Draw The labeled Level Curves;
figure(2)
[C,h] = contour(xx,yy,f(xx,yy),levels);
clabel(C,h);
title('Some Level curves of \(f(x,y) = y^2 - x^2\)','Color','red');
box off
grid on
axis equal;
set(gca,'Color','yellow');
axis([-5 5 -5 5]);
xlabel('x-indep','Color','blue'), ylabel('y-indep','Color','red');
colormap gray

% Draws The Surface \(z = y^2 - x^2\);
figure(1);
surf(u,v,f(u,v));
colormap gray;
rotate3d on;
axis equal;
set(gca,'Color','yellow');
axis([-5 5 -5 5 0 25]);
view([136 57]);
title('Hyperbolic Paraboloid', 'z = y^2 - x^2', 'Color','red');
xlabel('x-independent','Color','blue');
ylabel('y-independent','Color','red');
zlabel('z-dependent','Color','green');
Partial Derivatives

When we hold all but one of the independent variables of a function of several variables constant and differentiate with respect to that one variable, we get what is called a “partial” derivative. This discussion shows how partial derivatives arise and how to calculate partial derivatives by applying the familiar rules from differentiating functions of a single variable.

**Partial Derivative of a Function of Two Variables**

Suppose that we have a function $z = f(x, y)$ of independent variables $x$ and $y$ and whose “graph” is a surface in 3-space. If $(x_0, y_0)$ is a point in the domain of $f$, then the vertical plane $y = y_0$ ($0x + 1y + 0z = y_0$) will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0) = g(x)$ (the intersection of two surfaces is usually a curve). This curve is the graph of the function $z = g(x) = f(x, y_0)$ in the vertical plane $y = y_0$. The horizontal coordinates in this plane is $x$; the vertical coordinate is $z$. We define the partial derivative of $z = f(x, y)$ with respect to $x$ at the point $(x_0, y_0)$ as the ordinary derivative of $f(x, y_0)$ with respect to $x$ at $x = x_0$.

**Definition  Partial Derivative with Respect to $x$**

The **partial derivative of $f(x, y)$ with respect to $x$** at the point $(x_0, y_0)$ is

$$\frac{\partial f}{\partial x}(x_0, y_0) = g'(x_0) = \lim_{h \to 0} \left[ \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \right], \ h \neq 0$$

provided the limit exists.

**Notations for the partial derivative with respect to $x$:**

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_x(x_0, y_0), \quad \frac{\partial z}{\partial x}|_{(x_0, y_0)}, \quad f_x, \quad \frac{\partial f}{\partial x}, \quad z_x, \text{ or } \frac{\partial z}{\partial x}$$

**Geometric Interpretation of the partial derivative with respect to $x$**

The slope of the curve $z = g(x) = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ (on the surface $z = f(x, y)$) in the plane $y = y_0$ is the value of the partial derivative of $f$ with respect to $x$ at $(x_0, y_0)$. The tangent line to the curve at $P$ is the line with equation $z = f(x_0, y_0) = f_x(x_0, y_0)(x - x_0)$ in the plane $y = y_0$ that passes through $P$ with slope $m = f_x(x_0, y_0)$. The partial derivative $\frac{\partial f}{\partial x}$ at $(x_0, y_0)$ gives the **rate of change of $f$ with respect to $x$ when $y$ is held fixed at the value $y = y_0$**. This is the rate of change of $f$ in a direction parallel to the unit vector $\hat{i}$ at $(x_0, y_0)$.
The definition of the partial derivative of \( f(x, y) \) with respect to \( y \) at a point \((x_0, y_0)\) is similar to the definition of the partial derivative of \( f \) with respect to \( x \). We hold the variable \( x \) fixed at the value \( x = x_0 \) and take the ordinary derivative of \( f(x_0, y) \) with respect to the variable \( y \).

**Definition** Partial Derivative with Respect to \( y \)

The partial derivative of \( f(x, y) \) with respect to \( y \) at the point \((x_0, y_0)\) is

\[
\frac{\partial f}{\partial y}(x_0, y_0) = h'(y_0) = \lim_{{h \to 0}} \left[ \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \right], \quad h \neq 0
\]

provided the limit exists.

**Notations for the partial derivative with respect to \( x \):**

\[
\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_y(x_0, y_0), \quad \frac{\partial z}{\partial y} \big|_{(x_0, y_0)} \quad \frac{\partial f}{\partial y}, \quad z_y, \quad \text{or} \quad \frac{\partial z}{\partial y}
\]

**Geometric Interpretation of the partial derivative with respect to \( y \)**

The slope of the curve \( z = h(y) = f(x_0, y) \) at the point \( P(x_0, y_0, f(x_0, y_0)) \) (on the surface \( z = f(x, y) \)) in the plane \( x = x_0 \) is the value of the partial derivative of \( f \) with respect to the variable \( y \) at \((x_0, y_0)\). The tangent line to the curve at \( P \) is the line with equation \( z - f(x_0, y_0) = f_y(x_0, y_0)(y - y_0) \) in the plane \( x = x_0 \) that passes through \( P \) with slope \( m = f_y(x_0, y_0) \). The partial derivative \( \frac{\partial f}{\partial y} \) at \((x_0, y_0)\) gives the rate of change of \( f \) with respect to \( y \) when \( x \) is held fixed at the value \( x = x_0 \). This is the rate of change of \( f \) in a direction parallel to the unit vector \( \hat{j} \) at \((x_0, y_0)\).

**Example -1.6** Finding Partial Derivatives at a Point Find the values of \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) at the point \((4, -5)\) if

\[
z = f(x, y) = x^2 + 3xy + y - 1.
\]

**Solution** To find \( \frac{\partial f}{\partial x} \), we treat \( y \) as a constant and differentiate with respect to \( x \):

\[
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.
\]

The value of \( \frac{\partial f}{\partial x} \) at \((4, -5)\) is

\[
\frac{\partial f}{\partial x}(4, -5) = 2(4) + 3(-5) = -7.
\]
To find \( \frac{\partial f}{\partial y} \), we treat \( x \) as a constant and differentiate with respect to \( y \):

\[
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.
\]

The value of \( \frac{\partial f}{\partial x} \) at \((4, -5)\) is

\[
\frac{\partial f}{\partial y}(4, -5) = 3(4) + 1 = 13.
\]

**Example -1.7  FINDING A PARTIAL DERIVATIVE AS A FUNCTION** Find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) if \( f(x, y) = y \sin xy \).

**Solution**

For \( \frac{\partial f}{\partial x} \) we treat \( y \) as a constant and \( f \) as a constant times \( \sin xy \):

\[
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y \sin xy) = y \frac{\partial}{\partial x}(\sin xy) = y^2 \cos xy.
\]

For \( \frac{\partial f}{\partial y} \) we treat \( x \) as a constant and \( f \) as a product of \( y \) and \( \sin xy \):

\[
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y}(\sin xy) + (\sin xy) \frac{\partial}{\partial y}(y) = xy \cos xy + \sin xy.
\]

**Example -1.8  PARTIAL DERIVATIVES MAY BE DIFFERENT FUNCTIONS** Find \( f_x \) and \( f_y \) if

\[
f(x, y) = \frac{2y}{y + \cos x}.
\]

**Solution**

We treat \( f \) as a quotient. With \( y \) held constant, we get

\[
f_x = \frac{\frac{\partial}{\partial x} \left( \frac{2y}{y + \cos x} \right)}{\left( \frac{2y}{y + \cos x} \right)} = \frac{(y + \cos x) \frac{\partial}{\partial x}(2y) - 2y \frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.
\]
We treat \( f \) as a quotient. With \( x \) held constant, we get
\[
f_y = \frac{\partial}{\partial y} \left( \frac{2y}{y + \cos x} \right) = \frac{(y + \cos x) \frac{\partial}{\partial y} (2y) - 2y \frac{\partial}{\partial y} (y + \cos x)}{(y + \cos x)^2} = \frac{2 \cos x}{(y + \cos x)^2}.
\]

**Example -1.9** **Implicit Partial Differentiation** Find \( \partial z/\partial x \) if the equation
\[
yz - \ln z - x - y = 0
\]
defines \( z = f(x, y) \) as a function of the two independent variables \( x \) and \( y \) and the partial derivatives of \( f \) exists.

**Solution** The equation \( yz - \ln z - x - y = 0 \) which defines \( z = f(x, y) \) as a function means that we can replace \( z \) by \( f(x, y) \) everywhere \( z \) appears to get:
\[
y f(x, y) - \ln(f(x, y)) - x - y = 0. \tag{1}
\]
Treating \( y \) as constant and then differentiating (1) with respect to \( x \) on both sides:
\[
\frac{\partial}{\partial x} (y f(x, y) - \ln(f(x, y)) - x - y) = y f_x(x, y) - \frac{f_x(x, y)}{f(x, y)} - 1 = 0. \tag{2}
\]
Solving (2) for \( f_x(x, y) \) yields:
\[
f_x(x, y) = \frac{1}{y - \frac{1}{f(x, y)}} \text{ or equivalently, that } \frac{\partial z}{\partial x} = \frac{1}{y - \frac{1}{z}} = \frac{z}{yz - 1}.
\]

**Example -1.10** **A Function of Three Variables** If \( x \), \( y \), and \( z \) are independent variables and
\[
f(x, y, z) = x \sin(y + 3z),
\]
then
\[
\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} [x \sin(y + 3z)] = x \frac{\partial}{\partial z} (\sin(y + 3z)) = x \cos(y + 3z) \frac{\partial}{\partial z} (y + 3z) = 3x \cos(y + 3z).
\]
A function $f(x, y)$ can have partial derivatives with respect to both $x$ and $y$ at a point without being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. If the partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at $(x_0, y_0)$, however, then $f$ is continuous at $(x_0, y_0)$, as we see in the next lesson.

**Example -1.11** Partial Derivatives Exist, But $f$ Discontinuous

Let

$$f(x, y) = \begin{cases} 
0, & xy \neq 0 \\
1, & xy = 0 
\end{cases}$$

(a) Find the limit of $f$ as $(x, y)$ approaches $(0, 0)$ along the line $y = x$.

(b) Prove that $f$ is not continuous at the origin.

(c) Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

**Solution**

(a) Since $f(x, y)$ is constantly zero along the path $y = x$ (except at the origin), we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} 0 = 0.$$  

(b) Since $f(0,0) = 1$, the limit in part (a) proves that $f$ is not continuous at $(0,0)$.

(c) To find $\partial f/\partial x$ at $(0,0)$, we hold $y$ fixed at $y = 0$. Then $f(x, y) = f(x, 0) = 1$ for all $x$ (The graph is parallel to the $y$-axis at $z = 1$). The slope of this line at any $x$ is $\partial f/\partial x = 0$. In particular, $\partial f/\partial x = 0$ at $(0,0)$. Similarly, $\partial f/\partial y$ is the slope of the line $f(x, y) = f(0, y) = 1$ for all $y$, so $\partial f/\partial y = 0$ at $(0,0)$.

While it is still true in higher dimension that differentiability at a point implies continuity, the example just above suggests that we need a stronger requirement for differentiability in higher dimensions than the mere existence of the partial derivatives. We define differentiability for functions of two variables at the end of this lesson.
SECOND-ORDER PARTIAL DERIVATIVES

When we differentiate a function \( f(x, y) \) twice, we produce its second-order derivatives. These derivatives are usually denoted by:

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = f_{xx},
\]

\[
\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = f_{yy},
\]

\[
\frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = f_{yx},
\]

\[
\frac{\partial^2 f}{\partial y \, \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = f_{xy}.
\]

**Example 1.12 Finding Second-Order Partial Derivatives** If \( f(x, y) = x \cos y + y e^x \), find \( f_{xx}, f_{xy}, f_{yy}, \) and \( f_{yx} \).

**Solution**

\[
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + y e^x) = \cos y + y e^x; \quad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + y e^x) = -x \sin y + e^x
\]

So

\[
f_{xy} = \frac{\partial^2 f}{\partial y \, \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y}(\cos y + y e^x) = -\sin y + e^x
\]

\[
f_{yx} = \frac{\partial^2 f}{\partial x \, \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x}(-x \sin y + e^x) = -\sin y + e^x
\]

\[
f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y}(-x \sin y + e^x) = -x \cos y.
\]

\[
f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x}(\cos y + y e^x) = y e^x
\]
Partial Derivatives

The Mixed Derivative Theorem (Clairaut)

You may have noticed that the “mixed” second-order partial derivatives
\[ \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} \]
in the previous example were equal. This was no accident. They must be equal whenever \( f, f_x, f_y, f_{xy}, \) and \( f_{yx} \) are continuous, as stated in the following theorem:

Clairaut’s Theorem (Equality of Mixed Partials) Suppose that \( f(x, y) \) is a function defined throughout some open disk, \( D \subseteq \mathbb{R}^2 \) centered on a point \( (x_0, y_0) \). If \( f(x, y) \) together with its partial derivatives \( f_x, f_y, f_{xy}, \) and \( f_{yx} \) are all continuous at \( (x_0, y_0) \), then

\[ f_{xy}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = f_{yx}(x_0, y_0) \]

Example -1.13 Choosing the Order of Differentiation Suppose that you are given the function \( w \) defined by

\[ w = x y + \frac{e^y}{y^2 + 1} \]

and you are asked to find \( \frac{\partial^2 w}{\partial x \partial y} \).

Solution The symbol \( \frac{\partial^2 w}{\partial x \partial y} \) tells us to first differentiate \( w \) with respect to the variable \( y \) (while \( x \) is held constant) and then differentiate the resulting expression with respect to \( x \) (while the variable \( y \) is held constant). However, we see that \( w \) and its first and second-order partial derivatives are everywhere defined and everywhere continuous on \( \mathbb{R}^2 \) (why?). Clairaut’s theorem gives us the optional of changing the order in which we differentiate provided that less work results: Thus, if we first differentiate with respect to \( x \) and then differentiate with respect to \( y \) we obtain our answer far more easier as follows:

Since

\[ \frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \left( x y + \frac{e^y}{y^2 + 1} \right) = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} \right) = \frac{\partial}{\partial y} (y) = 1 = \frac{\partial^2 w}{\partial x \partial y}, \]

by Clairaut’s Theorem. We are in for more work if we differentiate first with respect to \( y \). (Give it a try!)
Using MATLAB 7 to Find partial derivatives of Functions of two variables

```matlab
clear all;
close all;
clc;
syms x y z;  % make x and y symbolic variables.
w = x * y + (exp(y)/(1 + y^2));
wy = diff(w, y);  % the first-partial derivative expression with respect to x.
wyx = diff(wy, x);  % the first-partial derivative expression with respect to y.
wyx = diff(diff(w, y), x)  % takes a mixed second-order derivative.
wxy = diff(diff(w, x), y)  % The partial derivative expression turned into functions.
```

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

\[
\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}
\]

\[
\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx},
\]

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as the derivatives through the order in question are continuous.
Differentiability, Differentials, and Local Linearity

In this discussion we will extend the notion of differentiability to functions of two or three variables. Our definition of differentiability will be based on the idea that a function is differentiable at a point provided it can be very closely approximated by a linear function near that point. In the process, we will expand the concept of a “differential” to functions of more than one variable and define the “local linear approximation” of a function.

What do we mean when we say that a function of two or three variables is differentiable at a point?

Recall that function $f$ of one variable is called differentiable at $x_0$ if it has a derivative at $x_0$, that is, if the limit

$$f'(x_0) = \lim_{\Delta x \to 0} \left[ \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \right]$$

exists. As a consequence of (1) a differentiable function enjoys a number of other important properties:

1) The graph of $y = f(x)$ has nonvertical tangent line at the point $(x_0, f(x_0))$;
2) $f$ may be closely approximated by a linear function near $x_0$;
3) $f$ is continuous at $x_0$.

Our primary objective in this discussion is to extend the notion of differentiability to function of two or three variables in such a way that the natural analogs of these properties hold. For example, if a function $f(x, y)$ of two variables is differentiable at a point $(x_0, y_0)$, we want it to be the case that

1) The surface $z = f(x, y)$ has nonvertical tangent plane at the point $(x_0, y_0, f(x_0))$;
2) The values of $f$ at points $(x_0, y_0)$ can be very closely approximated by the values of a linear function.
3) $f$ is continuous at $(x_0, y_0)$.

One could reasonably conjecture that a function $f$ of two or three variables should be called differentiable at a point if all the first-order partial derivatives of the function exist at that point. Unfortunately, this condition is not strong enough to guarantee that the properties hold. For instance, we saw in Example 0.6 that the mere existence of both first-order partial derivatives for a function is not sufficient to guarantee the continuity of the function. To determine what else we should include in our definition, it will be helpful to reexamine one other consequence of differentiability for a single-variable function $f(x)$. Suppose that $y = f(x)$ is a differentiable function at $x_0$ and let

$$\Delta f = f(x_0 + \Delta x) - f(x_0)$$

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denote the change in $f$ that corresponds to the change, $\Delta x$ in $x$ from $x_0$ to $x_0 + \Delta x$. We learned in the calculus that

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$$

provided $\Delta x$ is close to 0. In fact, for $\Delta x$ close to 0 the error,

$$\Delta f - f'(x_0)\Delta x = f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x$$

in this approximation will have magnitude much smaller than the magnitude of $\Delta x$ (the distance between $x_0$ and $x_0 + \Delta x$)

$$\lim_{\Delta x \to 0} \frac{\Delta f - f'(x_0)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \left[ \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - f'(x_0) \right] = f'(x_0) - f'(x_0) = 0.$$ 

Since $|\Delta x| = |x_0 + \Delta x - x_0|$, we see that when the two points $x_0$ and $x_0 + \Delta x$ are close together, the magnitude of the error, $|\Delta f - f'(x_0)\Delta x|$ in the approximation will be much smaller than $|\Delta x|$, i.e., For any given $\varepsilon > 0$, we can find $\delta > 0$ such that $|\Delta f - f'(x_0)\Delta x| < \varepsilon |\Delta x|$ provided that $0 < |\Delta x| < \delta$. The extension of this idea to functions of two or three variables is the “extra ingredient” needed in our definition of differentiability for multivariable functions.

For a function $z = f(x, y)$ of two variables $x$ and $y$, the symbol $\Delta f$ (or $\Delta z$), called the **increment** of $f$, denotes the change in the value of $f(x, y)$ that results when $(x, y)$ varies from some initial position $(x_0, y_0)$ to some new position $(x_0 + \Delta x, y_0 + \Delta y)$; thus

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$ \hspace{1cm} (2)

Let us assume that both $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and (by analogy with one-variable) make the approximation

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$ \hspace{1cm} (3)

For $\Delta x$ and $\Delta y$ close to 0, we would like the error

$$\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y$$

in this approximation to be much smaller than the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between $(x_0, y_0)$ and $(x_0 + \Delta x, y_0 + \Delta y)$.

We can guarantee this by requiring that

$$\lim_{(\Delta x, \Delta y) \to (0, 0)} \frac{\Delta f - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

Based on these ideas, we can now give our definition of differentiability for functions of two variables
**Definition**  A function $f$ of two variables, say $x$ and $y$ is said to be **differentiable** at $(x_0, y_0)$ provided that:

a) $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and

b) 
\[
\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta f - f_x(x_0, y_0) \Delta x - f_y(x_0, y_0) \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0 \tag{4}
\]

**Example -1.14**  Use the definition above to show prove that $f(x, y) = \frac{1}{3} \pi x^2 y$ is differentiable at $(0,0)$.

**Solution**  The increment is
\[
\Delta f = f(0 + \Delta x, 0 + \Delta y) - f(0, 0) = \frac{1}{3} \pi (\Delta x)^2 (\Delta y)
\]

Since $f_x(x, y) = \frac{2}{3} \pi xy$ and $f_y(x, y) = \frac{1}{3} \pi x^2$, we have $f_x(0, 0) = f_y(0, 0) = 0$, and (4) becomes
\[
\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\frac{1}{3} \pi (\Delta x)^2 (\Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0
\]

you may want to verify this by observing that
\[
0 \leq \left| \frac{(\Delta x)^2 (\Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right| \leq (\Delta x)^2
\]
for all $(\Delta x, \Delta y) \neq (0,0)$.

\[
\frac{1}{3} \pi \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{(\Delta x)^2 \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0.
\]

Thus, $f$ is differentiable at $(0,0)$.

We now derive an important consequence of limit (4). Define a function $\epsilon = \epsilon(\Delta x, \Delta y)$ of two variables $\Delta x$ and $\Delta y$ by
\[
\epsilon = \epsilon(\Delta x, \Delta y) = \begin{cases} 
\frac{\Delta f - f_x(x_0, y_0) \Delta x - f_y(x_0, y_0) \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}, & (\Delta x, \Delta y) \neq (0,0) \\
0, & (\Delta x, \Delta y) = (0,0)
\end{cases}
\]

Equation (4) then implies that
\[
\lim_{(\Delta x, \Delta y) \to (0,0)} \epsilon(\Delta x, \Delta y) = 0
\]
Furthermore, it immediately follows from the definition of \( \epsilon \) that

\[ \Delta f = f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon \sqrt{(\Delta x)^2 + (\Delta y)^2} \]  

(5)

In other words, if \( f \) is differentiable at \((x_0, y_0)\), then \( \Delta f \) may be expressed as shown in (5), where \( \epsilon \to 0 \) as \((\Delta x, \Delta y) \to (0,0)\) and where \( \epsilon = 0 \) if \((\Delta x, \Delta y) = (0,0)\).

For functions of three variables we have an analogous definition of differentiability in terms of the increment

\[ \Delta f = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0) \]

\[ \lim_{(\Delta x, \Delta y, \Delta z) \to (0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0) \Delta x - f_y(x_0, y_0, z_0) \Delta y - f_z(x_0, y_0, z_0) \Delta z}{\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}} = 0 \]  

(6)

In a manner similar to the two-variable case, we can express the limit (6) in terms of a function \( \epsilon(\Delta x, \Delta y, \Delta z) \) that vanishes at \((\Delta x, \Delta y, \Delta z) = (0,0,0)\) and is continuous there.

If a function \( f \) of two variables is differentiable at each point of an open region \( R \) in the \( xy \)-plane, then we say that \( f \) is differentiable on \( R \); and if \( f \) is differentiable at every point in the \( xy \)-plane, then we say that \( f \) is differentiable everywhere. For a function of three variables we have corresponding theorems and conventions.

**Differentiability and Continuity**

Recall that we want a function to be continuous at every point at which it is differentiable. The next result shows this to be the case.

**Theorem** If a function is differentiable at a point, then it is continuous at that point.

**(Why?)** Let us suppose that \( f(x, y) \) is a function of two variables \( x \) and \( y \) which is differentiable at a point \((x_0, y_0)\) in some open region \( D \subseteq \mathbb{R}^2 \). Then we must show that

\[ \lim_{(x,y) \to (x_0,y_0)} f(x,y) = f(x_0,y_0) \]
Let \( x = x_0 + \Delta x \) and \( y = y_0 + \Delta y \), is equivalent to showing that

\[
\lim_{(\Delta x, \Delta y) \to (0,0)} f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0)
\]

which is equivalent to

\[
\lim_{(\Delta x, \Delta y) \to (0,0)} \Delta f = 0.
\]

However, from Equation (5)

\[
\lim_{(\Delta x, \Delta y) \to (0,0)} \Delta f = \lim_{(\Delta x, \Delta y) \to (0,0)} \left( f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y + \epsilon(\Delta x, \Delta y) \sqrt{(\Delta x)^2 + (\Delta y)^2} \right)
\]

\[
= 0 + 0 + 0(0) = 0
\]

It can be difficult to verify that a function is differentiable at a point directly from the definition. The next theorem, whose proof is usually studied in more advanced courses, provides simple conditions for a function to be differentiable at a point.

**Theorem [Differentiability the easy way]** If \( f \) is a function of two variables \( x \) and \( y \) defined in some open disk centered at \((x_0, y_0)\) and if \( f_x(x_0, y_0) \) and \( f_y(x_0, y_0) \) exist and is continuous at \((x_0, y_0)\), then \( f \) is differentiable at \((x_0, y_0)\).

For example, consider the function \( f(x, y, z) = x + yz \)

Since \( f_x(x, y, z) = 1, f_y(x, y, z) = z, \) and \( f_z(x, y, z) = y \) are defined and continuous everywhere, we conclude from the (three variable extension of the above theorem) that \( f \) is differentiable everywhere.

**Differentials**

As with the one-variable case, the approximations

\[
f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y
\]

for a function of two variables and the approximation

\[
f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) \approx f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0) \Delta x + f_y(x_0, y_0, z_0) \Delta y + f_z(x_0, y_0, z_0) \Delta z
\]

For a function of three variables have a convenient formulation in the language of differentials. If \( z = f(x, y) \) is differentiable at a point \((x_0, y_0)\), we let

\[
dz = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy
\]

(7)
denote a new function with dependent variable $dz$ and independent variables $dx$ and $dy$. We refer to this function (also denoted $df$) as the total differential of $z$ at $(x_0, y_0)$ or as the total differential of $f$ at $(x_0, y_0)$. Similarly, for a function $w = f(x, y, z)$ of three variables we have the total differential of $w$ at $(x_0, y_0, z_0)$,

$$dw = f_x(x_0, y_0, z_0)dx + f_y(x_0, y_0, z_0)dy + f_z(x_0, y_0, z_0)dz \quad \text{(8)}$$

which is also referred to as the total differential of $f$ at $(x_0, y_0, z_0)$. It is common practice to omit the subscripts and write Equations (7) and (8) as

$$dz = f_x(x, y)dx + f_y(x, y)dy \quad \text{(9)}$$

and

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz \quad \text{(10)}$$

In the two-variable case, the approximation

$$\Delta f \approx f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y$$

can be written in the form

$$\Delta f \approx df \quad \text{(11)}$$

for $dx = \Delta x$ and $dy = \Delta y$. Equivalently, we can write approximation (11) as

$$\Delta z \approx dz \quad \text{(12)}$$

In other words, we can estimate the change $\Delta z$ in $z$ by the value of the differential $dz$ where $dx$ is the change in $x$ and $dy$ is the change in $y$. Furthermore, it follows from (4) that if $\Delta x$ and $\Delta y$ are close to 0, then the magnitude of the error in approximation (12) will be much smaller than the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ between $(x_0, y_0)$ and $(x_0 + \Delta x, y_0 + \Delta y)$.

**Example -1.15** Use differentials to approximate the change in $z = xy^2$ from its value at $(0.5, 1.0)$ to its value at $(0.503, 1.004)$. Compare the magnitude of the error in this approximation with the distance between the points $(0.5, 1.0)$ and $(0.503, 1.004)$

**Solution** For $z = f(x, y) = xy^2$ we have $dz = y^2 dx + 2xy dy$. Evaluating this differential at $(x, y) = (0.5, 1.0)$, $dx = \Delta x = 0.503 - 0.5 = 0.003$, and $dy = \Delta y = 1.004 - 1.0 = 0.004$ yields

$$dz = \Delta x \cdot y^2 = 0.003(1.0)^2 = 0.003$$

Since $z = 0.5$ at $(x, y) = (0.5, 1.0)$ and $z = 0.507032048$ at $(x, y) = (0.503, 1.004)$, we have

$$\Delta z = 0.507032048 - 0.5 = 0.007032048$$
and the error, \( E = dz - \Delta z \) in approximating \( \Delta z \) by \( dz \) has magnitude, \(| E | \) given by

\[
| dz - \Delta z | = |0.007 - 0.007032048| = 0.000032048
\]

Since the distance between \((0.5, 1.0)\) and \((0.503, 1.004)\) is

\[
\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(0.003)^2 + (0.004)^2} = \sqrt{0.000025} = 0.005
\]

we have

\[
\frac{| dz - \Delta z |}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{0.000032048}{0.005} = 0.00064096 < \frac{1}{150}
\]

Thus, the magnitude of the error in our approximation of \( \Delta z \) by \( dz \) is less than \( 1/150 \) of the distance between \((0.5, 1.0)\) and \((0.503, 1.004)\).

With the appropriate change in notation, the preceding analysis can be extended to functions of three or more variables.

**Example -1.16** Predicting Measurement Error

The length, width, and height of a rectangular box are measured with an error of at most 5%. Use differentials to estimate the maximum percentage error that results if these measured quantities are used to calculate the diagonal of the box.

**Solution** The diagonal \( F \) of a rectangular box with length \( x \), width \( y \), and height \( z \) is given by

\[
F = F(x, y, z) = \sqrt{x^2 + y^2 + z^2}; \quad \text{Dom}(F) = \text{first octant}
\]

Let \( x_0, y_0, z_0, \) and \( F_0 = F(x_0, y_0, z_0) = \sqrt{x_0^2 + y_0^2 + z_0^2} \) denote the actual values of the length, width, height, and diagonal of the box respectively. To estimate \( \Delta F \), we use the total differential \( dF \) of \( F \) at \((x_0, y_0, z_0)\) is given by

\[
dF = F_x(x_0, y_0, z_0) \, dx + F_y(x_0, y_0, z_0) \, dy + F_z(x_0, y_0, z_0) \, dz
\]

\[
= \frac{x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}} \, dx + \frac{y_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}} \, dy + \frac{z_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}} \, dz
\]

\[
= \frac{x_0}{F_0} \, dx + \frac{y_0}{F_0} \, dy + \frac{z_0}{F_0} \, dz
\]

If \( x, y, \) and \( z \) are the measured and computed values of length, width, height, and diagonal, respectively, then \( \Delta x = x - x_0, \Delta y = y - y_0, \Delta z = z - z_0, \Delta F = F - F_0 \) and

\[
\left| \frac{\Delta x}{x_0} \right| \leq 0.05, \quad \left| \frac{\Delta y}{y_0} \right| \leq 0.05, \quad \left| \frac{\Delta z}{z_0} \right| \leq 0.05.
\]

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We are seeking to estimate for the maximum size of $\Delta F/F_0$. So,

$$\frac{\Delta F}{F_0} \approx \frac{dF}{F_0} = \frac{1}{F_0^2} \left[ x_0 \Delta x + y_0 \Delta y + z_0 \Delta z \right]$$

Since

$$\left| \frac{dF}{F_0} \right| = \frac{1}{F_0^2} \left| x_0^2 \frac{\Delta x}{x_0} + y_0^2 \frac{\Delta y}{y_0} + z_0^2 \frac{\Delta z}{z_0} \right|$$

$$\leq \frac{1}{F_0^2} \left( x_0^2 \frac{\Delta x}{x_0} + y_0^2 \frac{\Delta y}{y_0} + z_0^2 \frac{\Delta z}{z_0} \right)$$

$$\leq \frac{1}{F_0^2} (x_0^2(0.05) + y_0^2(0.05) + z_0^2(0.05)) = \frac{1}{F_0^2} (0.05 F_0^2)$$

$$= 0.05.$$

We estimate the maximum percentage error in the calculation of the diagonal of the box to be 5%.

**Example -1.17** The volume $V = \frac{1}{3} \pi r^2 h$ of a right circular upper cone is to be calculated from measured values of $r$ and $h$. Suppose that $r$ is measured with an error of no more than 2% and $h$ with an error of no more than 0.5%. Estimate the resulting possible percentage error in the calculation of $V$.

**Solution** We are told that

$$\left| \frac{dr}{r_0} \right| \leq 0.02 \quad \text{and} \quad \left| \frac{dh}{h_0} \right| \leq 0.005.$$ 

Since

$$\frac{\Delta V}{V_0} \approx \frac{dV}{V_0} = \frac{\frac{2}{3} \pi r_0 h_0 \, dr}{\frac{1}{3} \pi r_0^2 h_0} + \frac{\frac{1}{3} \pi r_0^2 \, dh}{\frac{1}{3} \pi r_0^2 h_0} = \frac{2 \, dr}{r_0} + \frac{dh}{h_0},$$

where $r_0, h_0,$ and $V_0$ are the actual values of radius, height, and Volume respectively. Thus, we have

$$\left| \frac{dV}{V_0} \right| = \left| 2 \frac{dr}{r_0} + \frac{dh}{h_0} \right|$$

$$\leq 2 \left| \frac{dr}{r_0} \right| + \left| \frac{dh}{h_0} \right|$$

$$= 2(0.02) + 0.005$$

$$= 0.045.$$

We estimate the error in the volume calculation to be at most 4.5%.
In the problem above, it is quite conceivable that an error of 4.5% in the calculation of the volume may be unacceptable in quality control situations. What if quality control demands at most a 2% error in calculating the volume, then how accurately should we measure \( r \) and \( h \) so that we can calculate the volume \( V \) with an error of less than 2%? Questions like this one are difficult to answer because there is usually no single correct answer. Since

\[
\frac{dV}{V} = 2 \frac{dr}{r} + \frac{dh}{h}
\]

we see that \( dv/V \) is controlled by a combination of \( dr/r \) and \( dh/h \). If we can measure \( h \) with great accuracy, we might come out all right even if we are sloppy about measuring \( r \). On the other hand, our measurement of \( h \) might have so large a \( dh \) that the resulting \( dV/V \) would be too crude an estimate of \( \Delta V/V \) to be useful even if \( dr \) were zero.

What we do in such cases is: look for a reasonable square about the actual values \( (r_0, h_0) \) in which \( V \) will not vary by more than the allowed amount from \( V_0 = 1/3 \pi r_0 h_0 \). 

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Example -1.18  Controlling the Error

Find a reasonable square about the point \((r_0, h_0) = (6, 12)\) in which the value of \(V = \frac{1}{3} \pi r^2 h\) will not vary by more than \(\pm 0.1\).

Solution  We approximate the variation \(\Delta V\) by the differential

\[ dV = \frac{2}{3} \pi r h_0 \, dr + \frac{1}{3} \pi r_0^2 \, dh = \frac{2}{3} \pi (6)(12) \, dr + \frac{1}{3} \pi (6)^2 \, dh = 48 \pi \, dr + 12 \pi \, dh. \]

Since the region to which we are restricting our attention is a square, we may set \(dh = dr\) to get

\[ dV = 48 \pi \, dr + 12 \pi \, dr = 60 \pi \, dr. \]

We then ask, how small must we take \(dr\) to be sure that \(|dV|\) is no larger than 0.1? To answer, we start with the inequality

\[ |dV| \leq 0.1, \]

express \(dV\) in terms of \(dr\),

\[ |60 \pi \, dr| \leq 0.1, \]

and find a corresponding upper bound for \(dr\):

\[ |dr| \leq \frac{0.1}{60 \pi} \approx 5.3 \times 10^{-4}. \]

With \(dr = dh\), then, the square we want is described by the inequalities

\[ |\Delta r| = |r - 6| \leq 5.3 \times 10^{-4}, \quad |\Delta h| = |h - 12| \leq 5.3 \times 10^{-4}. \]

As long as \((r, h)\) stays in this square, we may expect \(|dV|\) to be less than or equal to 0.1 and we may expect \(|\Delta V|\) to be approximately the same size.

What do We Mean by Local Linear Approximation of a Function of two or three variable?

We now show that if a function \(f\) is differentiable at a point, it can be very closely approximated by a linear function near that point. For example, suppose that \(z = f(x, y)\) is differentiable at the point \((x_0, y_0)\). Then approximation (3) can be written in the form:

\[ f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y. \]

If we let \(x = x_0 + \Delta x\) and \(y = y_0 + \Delta y\), this approximation becomes

\[ f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad \text{(13)} \]

which yields a linear approximation of \(z = f(x, y)\). Since the error in this approximation is equal to the error in the approximation (3), we conclude that for \((x, y)\) close to \((x_0, y_0)\) the error in (13) will be much smaller than the distance between these two points. When \(z = f(x, y)\) is differentiable at \((x_0, y_0)\) we get

\[ L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad \text{(14)} \]

and refer to \(L(x, y)\) as the local linear approximation to \(f\) at \((x_0, y_0)\).
Example -1.19  Let \( L(x, y) \) be the local linear approximation to \( z = f(x, y) = \sqrt{x^2 + y^2} \) at the point \((3, 4)\). Compare the error in approximating

\[
f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2}
\]

by \( L(3.04, 3.98) \) with the distance between the points \((3, 4)\) and \((3.04, 3.98)\).

**Solution**  We have

\[
f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \quad \text{and} \quad f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}}
\]

with \( f_x(3, 4) = \frac{3}{5} \) and \( f_y(3, 4) = \frac{4}{5} \). Therefore, the local linear approximation to \( f \) at \((3, 4)\) is given by

\[
L(x, y) = 5 + \frac{3}{5}(x - 3) + \frac{4}{5}(y - 4).
\]

Consequently,

\[
f(3.04, 3.98) \approx L(3.04, 3.98) = 5 + \frac{3}{5}(0.04) + \frac{4}{5}(-0.02) = 5.008.
\]

Since

\[
f(3.04, 3.98) = \sqrt{(3.04)^2 + (3.98)^2} \approx 5.008193287
\]

the size of the error in the approximation is about

\[
| E(3.04, 3.98) | = | f(3.04, 3.98) - L(3.04, 3.98) | = | 5.008193287 - 5.008 | = 0.000193287.
\]

This is less than \( \frac{1}{200} \) of the distance

\[
\sqrt{(3.04 - 3)^2 + (3.98 - 4)^2} \approx 0.045
\]

between the points \((3, 4)\) and \((3.04, 3.98)\).

Similarly, for a function \( f(x, y, z) \) that is differentiable at \((x_0, y_0, z_0)\), the local linear approximation is

\[
L(x, y, z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \quad (15)
\]

We have formulated our definitions in this discussion in such a way that continuity and local linearity are consequences of differentiability. Later in another assignment, we will show that: **If a function \( f(x, y) \) is differentiable at a point \((x_0, y_0)\), then the graph of \( L(x, y) \) is a nonvertical tangent plane to the graph of \( f \) at the point \((x_0, y_0, f(x_0, y_0))\).**
Problem #1

a) Find $g(u(x, y), v(x, y))$ if $g(x, y) = y \sin(x^3 y)$, $u(x, y) = x^2 y^3$, and $v(x, y) = \pi x y$.

b) Sketch the domain of the functions: $f(x, y) = \ln(1 - x^2 - y^2)$ and $f(x, y) = \ln(x y)$ and $f(x, y) = \sqrt{x^2 + y^2 - 4}$ and $f(x, y) = \frac{1}{x - y^2}$. 
A point moves along the intersection of the plane $y = 3$ and the surface $z = \sqrt{29 - x^2 - y^2}$. At what rate is $z$ changing with respect to $x$ when the point is at $(4, 3, 2)$? Do this both ways (directly with limits and then without).
In complex analysis, a complex-valued function of a single complex-variable is a complex function of the form \( w = f(z) \) where \( w = u + iv \) and \( z = x + iy \) are complex numbers and \( u = u(x,y) \) and \( v = v(x,y) \) are the usual real-valued functions of two real variables \( x \) and \( y \) (\( u \) and \( v \) are the real and imaginary parts of \( w = f(z) \) respectively and \( x \) and \( y \) are real and imaginary parts of \( z \) respectively) A complex function \( w = f(z) \) is said to be differentiable on its complex domain if the following Cauchy-Riemann equations are satisfied:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]

Show that the complex functions

\[
w = f(z) = z^2 \quad \text{and} \quad w = g(z) = e^z
\]

are differentiable on its domain (the set of all complex numbers).

**Hint:** Expand \((x + iy)(x + iy)\) into its real and imaginary parts \( u \) and \( v \) respectively and see if they satisfy the Cauchy-Riemann equations. Similarly, expand \( e^{x+iy} = e^x e^{iy} \) and use the fact that: For any real number \( t \), it is known that

\[
e^{it} = \cos t + i \sin t; \quad \text{(called Euler’s Formula)}
\]
Section 3
Problem #4

a) In physics, the period $T$ (in seconds) of a simple pendulum (with small oscillations) is calculated using the formula

$$T = F(L, g) = 2\pi \sqrt{\frac{L}{g}},$$

where $L$ is the length of the pendulum and $g$ is the acceleration due to gravity. Suppose that measured values of $L$ and $g$ have errors of at most 0.5% and 0.1%, respectively. Use approximation by differentials to approximate the maximum percentage error in the calculated value of $T$.

b) Find a local linear approximation for $T$ at $(2, 9.8)$ and use it to approximate $T = F(2.2, 10)$.

c) Find a reasonable square $S$ centered on $(2, 9.8)$ such that for any $(L, g)$ in $S$, so that $T = F(L, g)$ can be approximated with an error of no more than $\pm 0.04$ seconds (when the oscillations are very small).
a) Show that the function of three variables $x$, $y$, and $z$ defined as:

$$f(x, y, z) = xy \sin z$$

is differentiable everywhere (Using a theorem)

b) For any point $(x_0, y_0, z_0)$ in 3-space, calculate

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0).$$

**Hint:** Use the fact that for $\Delta z$ near 0 we can replace $\cos(\Delta z)$ with 1 (i.e., $\cos(\Delta z) = 1$) and replace $\sin(\Delta z)$ with $\Delta z$ (i.e., $\sin(\Delta z) = \Delta z$).

c) See Example 1.14 and determine the function $\epsilon = \epsilon(\Delta x, \Delta y, \Delta z)$ using your computation in part (b).