Math 202 Final Exam Fall,2008

Part I Do all parts of the following six problems.

(1) Compute the derivative $\frac{dy}{dx}$ for each of the following (15 points) :

$$\text{(a)} \quad y = \ln(\frac{1}{x+1}).$$

$$\text{Ans:} \quad y' = -\frac{1}{x+1}.$$

$$\text{(b)} \quad y = \arcsin(\sqrt{x});$$

$$\text{Ans:} \quad y' = \frac{1}{2\sqrt{x(1-x)}}$$

$$\text{(c)} \quad y = x^{\cos(x)} + \cos(x);$$

$$\text{Ans:} \quad y' = x^{\cos(x)}[\frac{\cos(x)}{x} - \sin(x)\ln(x)] \; - \; \sin(x).$$

(2) Compute each of the following integrals(30 points):

$$\text{(a)} \int \frac{2x^2-4}{x^3-2x^2}\, dx;$$

$$\text{Ans:} \quad = \int \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x-2}\, dx \; = \; \ln|x| - \frac{2}{x} + \ln|x-2| + C.$$

$$\text{(b)} \int x\ln(x)\, dx;$$

$$\text{Ans:} \quad (\text{let } \; u = \ln(x) \; \text{ and } \; dv = xdx) \quad = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C.$$

$$\text{(c)} \quad \int \sqrt{4-x^2}\, dx;$$

$$\text{Ans:} \quad (\text{let } \; \cos(t) = x) \quad = -4\int \sin^2(t)\, dt \; = \; -2t + \sin(2t) + C$$

$$= \quad -2\arccos(\frac{x}{2}) + \frac{x\sqrt{4-x^2}}{2} + C.$$

(d) $\displaystyle\int_0^1 \arctan(x)\ dx;$

Ans: (let $u = \arctan(x)$ and $dv = dx$) $\displaystyle = x\arctan(x) - \frac{1}{2}\ln(1+x^2)\ |_0^1$

$$= \frac{\pi}{4} - \frac{\ln(2)}{2}$$

(e) $\displaystyle\int (\sin(x)+1)^2\ dx.$

Ans: $\displaystyle = \frac{3x}{2} - 2\cos(x) + \frac{\sin(2x)}{4} + C.$

(3) Compute each of the following limits (10 points):

(a) $\displaystyle\lim_{x\to\infty} \sqrt{x}\ e^{-x};$

Ans: $\displaystyle = \lim_{x\to\infty} \frac{1}{2\sqrt{x}\ e^{x}} = 0.$

(b) $\displaystyle\lim_{x\to 1^+} x^{1/(x-1)}.$

Ans: $= e$ because $\displaystyle\lim_{x\to 1^+} \frac{\ln(x)}{x-1} = 1.$

(4) The region R in first quadrant of the xy plane is bounded by the curves $y = \sin(x)$, $x = \pi/2$ and $y = 0$. Set up two integrals (method of washers and method of shells) for the volume of the solid obtained by rotating R around the line $y = 2$. Do not compute the value of the integrals(8 points)

Washers: $\displaystyle = \int_0^{\pi/2} \pi 2^2 - \pi(2-\sin(x))^2\ dx;$

Shells: $\displaystyle = \int_0^1 2\pi(2-y)(\frac{\pi}{2} - \arcsin(y))\ dy.$

(5) Sketch the curve given by the equation $r = 3+\sin(\theta)$ in polar coordinates, labeling the x and y intercepts, and compute the area it encloses. (7 points)

Area $\displaystyle = \frac{1}{2}\int_0^{2\pi} (3+\sin(\theta))^2\ d\theta = \frac{19\pi}{2}.$

Part II Do all parts of three out of the following four problems (10 points each)

(6) (a) A 16 pound pull extends a spring 6 inches (= one half of a foot). Compute the work done stretching the spring an <u>additional</u> foot.

$$k = 16 \div \frac{1}{2} \qquad \text{Work} = \int_{1/2}^{3/2} 32x dx = 32.$$

(b) Evaluate the integral or show it is divergent: $\int_0^1 \ln(t)\ dt$.

$$\lim_{M\to 0^+} \int_M^1 \ln(t)\ dt = \lim_{M\to 0^+} (\ t\ln(t) - t\ |_M^1\ )$$
$$= (-1) - (\lim_{M\to 0^+} \frac{\ln(t)}{1/t}) = -1.$$

(7) Calculate the arc-length of the section of the curve $y = \ln(\sec(x))$ between $x = 0$ and $x = \pi/4$. Leave your answer in terms of logs, but you should evaluate any trig functions which appear.

$$\text{Ans:} = \int_0^{\pi/4} \sqrt{1+\tan^2(x)}\ dx = \int_0^{\pi/4} \sec(x)\ dx$$
$$= \ln(|\sec(x) + \tan(x)|)\ |_0^{\pi/4} = \ln(\sqrt{2}+1).$$

(8) (a) A radioactive substance has a half-life of 12 years. Derive a formula for the amount left after t years if you begin with 400 pounds of the substance (Show your work). After how many years will you have 100 pounds left?

$$X = 400e^{-\frac{\ln(2)t}{12}}. \qquad \text{If } X = 100, \text{ then } t = \frac{12\ln(4)}{\ln(2)} = 24.$$

(b) Compute $\int e^{2x} \sin(x)\ dx$.

$$I = -e^{2x}\cos(x) + 2e^{2x}\sin(x) - 4I; \qquad I = -\frac{1}{5}e^{2x}\cos(x) + \frac{2}{5}e^{2x}\sin(x).$$

(9) (a) Draw a labeled sketch of the conic whose equation is $y^2 + 2y = 9x^2 + 35$. Identify which sort of conic it is.

$\frac{(y+1)^2}{36} - \frac{x^2}{4} = 1$ is a hyperbola with center $(0, -1)$ and foci $(0, -1 \pm \sqrt{40})$.

(b) Find all values of t such that the tangent line to the curve given parametrically by $x = t^2 + t, y = 3t + 3t^2 - t^3$ is parallel to the line given by $y = 2x + 3$.

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3 + 6t - 3t^2}{2t+1};$$

$$\frac{dy}{dx} = 2 \quad \text{when} \quad 3t^2 - 2t - 1 = 0 : \qquad t = 1, -\frac{1}{3}.$$

(10) A conical tank, sitting on the ground with point up, is 12 feet high and has 12 foot diameter at the base (see figure). It is filled with liquid that has a density of 96 pounds per cubic foot. Compute the work necessary to pump enough water out the top so that the height of the remaining water is 7 feet.

$$\text{Work} = \int_7^{12} (12 - y)\ 96\ \pi\ (\frac{12 - y}{2})^2\ dy$$

$$= 24\pi \int_0^5 u^3\ du = 3750\pi.$$