203 Fall 2008 Group Final Solutions

1a: Take cross products of the normal to get the direction of the line:

```
Cross[{2, -1, 1}, {1, 0, -3}]
```

```
{3, 7, 1}
```

Then parameterize the line using the point and the direction to get:

$x = 4 + 3\,t,\;\; y = \;-7\,t,\;\; z = \;-3 + t$

1b: The desired plane contains the normals of the two specifed plane, so its normal is the same cross product as in 1a. That gives an equation of the plane as $3\,(x - 4) + 7\,(y + 2) + (z - 3) = 0$

2a: the gradient is:

$$f = x^2\, y + y^2\, z$$

$$x^2\, y + y^2\, z$$

```
{D[f, x], D[f, y], D[f, z]}
```

$$\{2\,x\,y,\; x^2 + 2\,y\,z,\; y^2\}$$

2b: the gradient will give a normal vector, so evaluating at the point gives:

$$\{2\,x\,y,\; x^2 + 2\,y\,z,\; y^2\}\;\;/.\;\;\{x \to 2,\;\; y \to -1,\;\; z \to 3\}$$

```
{-4, -2, 1}
```

So an equation of the tangent plane is $-4\,(x - 2) + -2\,(y + 1) + (z - 3) = 0$

2c: we just want $D_v\,f = \;\nabla f \cdot v$ to be negative, so there are many correct answers. For example, moving in the direction given by i gives:

```
{-4, -2, 1}.{1, 0, 0}
```

```
-4
```

3: First, sketch the triangular region to reverse the order, giving an integral which goes easily by substitution:

$$\int_0^1 \int_0^x \sqrt{1 + x^2}\; dy\, dx$$

$$\frac{1}{3}\left(-1 + 2\,\sqrt{2}\right)$$

4a: Absolutely convergent by the ratio test
4b: Convergent by the alternating series test, and then absolutely convergent by the integral test:

$$\int_2^{\infty} \frac{1}{x\,\mathrm{Log}[x]^2}\, dx$$

$$\frac{1}{\mathrm{Log}[2]}$$

4c: Convergent by the alternating series test, and then divergent by limit comparison with $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ so it is conditionally convergent

5: use the ratio test to get convergence on the interval -2<x<2. For x=2, divergent by comparison with the harmonic series. For x=-2, convergent by the alternating series test.

6a: Taking the first partials and setting them to zero and solving gives:

```
T = x^3 + y^3 - 3 x y
```

$x^3 - 3\,x\,y + y^3$

```
Solve[{D[T, x] == 0, D[T, y] == 0}]
```

$\{\{x \to 0, y \to 0\}, \{x \to 1, y \to 1\}, \{x \to -(-1)^{1/3}, y \to (-1)^{2/3}\}, \{x \to (-1)^{2/3}, y \to -(-1)^{1/3}\}\}$

At the two points with real coordinates (0,0) and (1,1), we evalute the discriminant to get:

```
D[T, x, x] D[T, y, y] - D[T, x, y]^2
```

$-9 + 36\,x\,y$

```
-9 + 36 x y /. {x → 0, y → 0}
```

-9

```
-9 + 36 x y /. {x → 1, y → 1}
```

27

```
D[T, x, x] /. {x → 1, y → 1}
```

6

So there is a relative min at (1,1) and a saddle at (0,0).

6b: we consider the two points from part a) together with the edges:

```
T /. {x → 0, y → t}
```

t^3

```
Solve[D[T /. {x → 0, y → t}, t] == 0]
```

$\{\{t \to 0\}, \{t \to 0\}\}$

```
Solve[D[T /. {x → 2, y → t}, t] == 0]
```

$\{\{t \to -\sqrt{2}\}, \{t \to \sqrt{2}\}\}$

```
Solve[D[T /. {x → t, y → 0}, t] == 0]
```

$\{\{t \to 0\}, \{t \to 0\}\}$

```
Solve[D[T /. {x → t, y → 2}, t] == 0]
```

$\{\{t \to -\sqrt{2}\}, \{t \to \sqrt{2}\}\}$

That gives potential critical points at the four corners, the interior point (1,1) and the points $(\sqrt{2}, 2)$ and $(\sqrt{2}, 0)$. Checking all of them, we get:

```
T /. {x → 1, y → 1}
T /. {x → 0, y → 3}
T /. {x → 2, y → 3}
T /. {x → 2, y → 0}
T /. {x → 0, y → 0}
T /. {x → √2, y → 3}
T /. {x → √2, y → 0}
```

-1

27

17

8

0

$27 - 7\sqrt{2}$

$2\sqrt{2}$

So the hottest point is (0,3) with temperature 27.

7: set up the integral in polar, to get

$$\int_0^{2\pi}\int_0^{a} r\left(a - \sqrt{r^2}\right) dr\, d\theta$$

$$\frac{1}{3} a^2 \pi\,(3\,a - 2\,\mathrm{Abs}[a])$$

8. So f(x,y) is of the form f(r)= $f(x^2 + y^2)$. Using the chain rule, we get $f_x = 2\,x\,f_r$ and $f_y = 2\,y\,f_r$ and the desired result.

9a. Use the geometric series with $r = -t^2$ to getthe function as $1 - t^2 + t^4 - t^6 + \ldots$ which converges for $-1 < t < 1$.

9b. Integrating gives $t - \frac{t^3}{3} + \frac{t^5}{5} - \ldots$ and then evaluating gives $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3\,3^{3/2}} + \frac{1}{5\,3^{5/2}} - \ldots$

For the desired error, using the alternating series analysis, we want $1/k\,(\mathrm{Sqrt}[3])^{2k+1}$ to be less than .01, which happens when k=3, so the first 3 terms are sufficient

$$\mathrm{Table}\left[N\left[\frac{1}{(2\,k+1)\left(\sqrt{3}\right)^{2\,k+1}}\right], \{k, 1, 4\}\right]$$

```
{0.06415, 0.01283, 0.00305476, 0.000791976}
```

10. To find the mass, we integrate the density which is $\rho = \sqrt{x^2 + y^2} = r$ to get, using polar coordinates:

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^2 r\, r\, dr\, d\theta$$

$$\frac{8\pi}{3}$$

For the x-coordinate of the center of mass:

$$\text{xbar} = \frac{1}{\frac{8\pi}{3}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^2 r\,\text{Cos}[\theta]\, r\, r\, dr\, d\theta$$

$$\frac{3}{\pi}$$

$$\text{ybar} = \frac{1}{\frac{8\pi}{3}}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^2 r\,\text{Sin}[\theta]\, r\, r\, dr\, d\theta$$

$$0$$

11. Using the shadow technique and integrating $\sqrt{1 + f_x^{\,2} + f_y^{\,2}}$, we get:

$$f = \sqrt{9 - x^2 - y^2}$$

$$\sqrt{9 - x^2 - y^2}$$

$$\text{Simplify}\left[1 + D[f, x]^2 + D[f, y]^2\right]$$

$$-\frac{9}{-9 + x^2 + y^2}$$

Converting to polar and doubling (there is a top piece and a bottom piece):

$$2\int_0^{2\pi}\int_0^2 \sqrt{\frac{9}{9 - r^2}}\, r\, dr\, d\theta$$

$$-12\left(-3 + \sqrt{5}\right)\pi$$

12. a) changing to polar gives the quantity as $\frac{r^3 \cos(\theta)\sin(\theta)}{r^2}$ which goes to zero as r goes to zero, independant of θ. So the limit exists and is zero.

12. b) Taking derivatives and evaluating:

$$\text{Table}\left[D\left[\frac{1}{x}, \{x, i\}\right], \{i, 1, 6\}\right]$$

$$\left\{-\frac{1}{x^2}, \frac{2}{x^3}, -\frac{6}{x^4}, \frac{24}{x^5}, -\frac{120}{x^6}, \frac{720}{x^7}\right\}$$

which gives the series:

Series$\left[\frac{1}{x}, \{x, 2, 5\}\right]$

$$\frac{1}{2} - \frac{x-2}{4} + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \frac{1}{32}(x-2)^4 - \frac{1}{64}(x-2)^5 + O[x-2]^6$$

Knowing that e is between 2 and 3 gives an error estimate of:

$$R_5(e) = f^{(6)}\frac{(z)}{(5+1)!}(e-2)^6$$

which is less than if we let z=3 and note that $(e-2)^6 < 1$:

$$\left| R_5(e) \right| < \frac{\frac{720}{2^7}}{720} 1 = \frac{1}{2^7} = \frac{1}{128}$$

so the error would be less than .01