M203 Fall 2007 Exam solutions

Q1a: find a normal to the plane by taking cross product of two vectors in the plane.
vector along line: <2,3,4> point on line: (1,2,3), displacement vector from line to point:
<1,2,3>-<1,2,0>=<0,0,3> Cross product: <2,3,4> x <0,0,3> = <9,-6,0> giving an equation: 9 ( x-1) - 6 (y-2) + 0 (z-3) = 0

Q1b: Find a normal vector to the plane by putting into a standard form:
x+2y-z=4 gives normal vector <1,2,-1>. That gives parameterization:
x=1+t
y=2+2t
z=3-t

Q2a: find tangent vector by computing the derivative and evaluating it at the proper time
\[ r = \left\{ \sqrt{t}, \frac{4}{t}, \frac{t^2}{2} \right\} \]
\[ \frac{dr}{dt} = \left\{ \frac{1}{2 \sqrt{t}}, - \frac{4}{t^2}, t \right\} \]
\[ \frac{dr}{dt} \cdot t \rightarrow 4 \]
\[ \left\{ \frac{1}{4}, -\frac{1}{4}, 4 \right\} \]
gives the parameterization:
x = 2 + \frac{t}{4}
y = 1 - \frac{t}{4}
z = 8 + 4t

Q2b: find the gradient of the implicit description of the surface:
\[ \{ \nabla \left[ \frac{x^3}{z^2} + 4 (y - 1)^2, x \right], \nabla \left[ \frac{x^3}{z^2} + 4 (y - 1)^2, y \right], \nabla \left[ \frac{x^3}{z^2} + 4 (y - 1)^2, z \right] \} \]
\[ \{ \frac{3 x^2}{z^2}, 8 (-1 + y), -\frac{2 x^3}{z^3} \} \]
Evaluate at the point to get a normal vector:
\[ \left\{ \frac{3 x^2}{z^2}, 8 (-1 + y), -\frac{2 x^3}{z^3} \right\} \cdot (x \rightarrow 1, y \rightarrow 2, z \rightarrow -1) \]
\[ \{ 3, 8, 2 \} \]
gives the equation 3(x-1) + 8(y-2) + 2(z+1) = 0

Q3a: find the gradient at the point and use the method \( D_u f = \nabla f \cdot u \)
\[ f = \frac{x}{2x + 3y} \]
\[ \frac{\partial}{\partial x} f = \frac{2x}{(2x + 3y)^2} + \frac{1}{2x + 3y} \]
\[ \frac{\partial}{\partial y} f = -\frac{3x}{(2x + 3y)^2} \]

\[ \text{grad} = \{D[f, x], D[f, y]\} \]
\[ \{ -\frac{2x}{(2x + 3y)^2} + \frac{1}{2x + 3y}, -\frac{3x}{(2x + 3y)^2} \} \]
\[ \text{grad} \cdot \{x \to 2, \ y \to -1\} \]
\[ \{-3, \ -6\} \]

\[ u = \frac{-1 - 2, 1 + 1}{\sqrt{(-3)^2 + 2^2}} \]
\[ \{-3/\sqrt{13}, \ 2/\sqrt{13}\} \]

So the resulting directional derivative is:
\[ \{-3, \ -6\} \cdot \{-3/\sqrt{13}, \ 2/\sqrt{13}\} \]
\[ -\frac{3}{\sqrt{13}} \]

For the maximum, we take the length of the gradient at P:
\[ \text{Norm}[\text{grad} \cdot \{x \to 2, \ y \to -1\}] \]
\[ 3 \sqrt{5} \]

Q4a: easier as a type II region:
\[ \int_0^6 \int_{y/2} f(x, y) \, dx \, dy \]

Q4b: a parabolic sliver. The other order of integration gives:
\[ \int_0^4 \int_{y/2} \, f(x, y) \, dx \, dy \]

Q5: region outside a cylinder, inside a sphere above the x-y plane. Volume via cylindrical coordinates:
\[ \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4+r^2}} r \, dz \, dr \, dt \]
\[ 2 \sqrt{3} \pi \]

Q6a: diverges by the nth term test, as the limit is not zero:
\[ \lim_{n \to \infty} \left( \frac{n}{n+1} \right)^n = 1/e \]
Q6b: converges by the alternating series test. For the absolute value of the terms, it diverges by the integral test, so this series converges conditionally.

\[ \int_2^\infty \frac{1}{x \log(x)} \, dx \]

Integrate: idiv : Integral of \( \frac{1}{x \log(x)} \) does not converge on \( (2, \infty) \). More...

\[ \int_2^\infty \frac{1}{x \log(x)} \, dx \]

Q6c: Taking the absolute value and comparing with \( \sum \frac{1}{n^2} \) the convergent p-series gives that this converges absolutely. (The alternating series test cannot be used as the signs do not alternate.)

Q7: Using the ratio test, we get

\[ \text{Limit} \left[ \frac{\frac{(x+1)^{n+1}}{(n+1)^2 \cdot 2^{n+1}}}{\frac{x^{n+1}}{(n)^2 \cdot 2^{n}}} \right]_{n \to \infty} \]

\[ \left\{ \frac{1+x}{2} \right\} \]

which we want to have absolute value less than 1, giving that the series converges for \(-3 < x < 1\). For the endpoints, we compare with a p-series with p=2 and get that it converges at both endpoints, which gives the interval of convergence as \([-3, 1]\).

Q8: setting both derivatives equal to zero and solving gives three critical points:

\[ f = x y^2 - 2 xy + x^2 \]

\[ x^2 - 2xy + xy^2 \]

Solve\([\{D[f, x] = 0, D[f, y] = 0\}]\)

\[ \{ (x \to 0, y \to 0), (x \to 0, y \to 2), (x \to \frac{1}{2}, y \to 1) \} \]

\[ \text{disc} = D[f, x, x] \cdot D[f, y, y] - D[f, x, y]^2 \]

\[ 4x - (-2 + 2y)^2 \]

\[ \text{disc} / \cdot \{ (x \to 0, y \to 0), (x \to 0, y \to 2), (x \to \frac{1}{2}, y \to 1) \} \]

\[ \{-4, -4, 2\} \]

\[ D[f, x, x] / \cdot \{ x \to \frac{1}{2}, y \to 1 \} \]

\[ 2 \]

which gives saddles at (0,0) and (0,2) and a min at \((\frac{1}{2}, 1)\).

Q9: Setting up the triple integral gives:
Q10: We start with the power series for the geometric series for \( \frac{1}{1-x^2} = 1 - x^2 + x^4 \ldots \) and integrate to get \( \int \arctan(x) = \int 1 - x^2 + x^4 \ldots = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \) which has the general form for the \( n \)th term as \( a_n = (-1)^{n+1} \frac{x^{2n+1}}{2n+1} \), starting at \( n=1 \).

To control the error, we note since the series is alternating, the error is no more than the subsequent term, so we need the absolute value of \( a_{n+1} < \frac{1}{10^4} \). The second term is \( \frac{10^{-3}}{3} = \frac{1}{3000} \) which is not small enough, and the third term is \( \frac{1}{5 \cdot 10^5} \) which is small enough, so the first two terms suffice.

Q11a: we approach along two lines. The horizontal line \( y=0 \) gives \( \frac{\pi}{2} \) which has limit 1, and the vertical line \( x=0 \) gives limit 0, so the limit cannot exist.

Q11b: completing the square twice to put this into standard form gives the equation:

\[-(x-1)^2 - y^2 + (z+1)^2 = 1\]

which is a hyperboloid of two sheets, opening in directions parallel to the z-axis, with vertices at \((1,0,0)\) and \((1,0,-2)\).

Q12: finding the surface area, we compute the integral of \( \sqrt{1 + f_x^2 + f_y^2} \) over the shadow of the region to get:

\[
\begin{align*}
  f_x &= D[4 - x^2 - y^2, x] \\
  f_y &= D[4 - x^2 - y^2, y] \\
  -2x \\
  -2y \\
  \sqrt{1 + f_x^2 + f_y^2} \\
  \sqrt{1 + 4x^2 + 4y^2}
\end{align*}
\]

converting to polar, we get for the surface area:

\[
\int_0^{2\pi} \left( \int_0^1 \sqrt{1 + 4r^2} \, r \, dr \right) \, d\theta
\]

\[\frac{\pi}{6} + \frac{13 \sqrt{13} \pi}{6}\]