MATH 203 Final Exam
December 18, 2008

Instructions: Complete every question in Part I (Questions 1-7.) Complete three of the five questions in Part II (Questions 8-12). Each question is worth 10 points.

No calculators or other electronic devices may be used during the examination. Answers may be left in terms of $\sqrt{7}$, $\sin(3)$, π etc. Proctors will confiscate any devices observed in the examination room. Give reasons for all answers. This exam will last 2 hours and 15 minutes. Good luck!

PART I: Answer all parts of Questions 1-7. Each question is worth 10 points

1. (a) Parameterize the line that passes through the point $(4, 0, -3)$ which is parallel to the intersection of the plane $2x - y + z = 2$ and the plane $x - 3z = 4$

   (b) Find an equation of the plane perpendicular to the planes $2x - y + z = 2$ and $x - 3z = 4$ that passes through $(4, -2, 3)$.

2. Let $f(x, y, z) = x^2y + y^2z$.

   (a) Evaluate ∇f.

   (b) Find an equation of the tangent plane to the surface $x^2y + y^2z = -1$ at the point $(2, -1, 3)$.

   (c) Find a unit vector $\vec{v}$ such that $f(x, y, z)$ has a negative rate of change at $(2, -1, 3)$ in the direction $\vec{v}$.

3. Evaluate the integral $\int_0^1 \int_y^1 \sqrt{1 + x^2}\, dx\, dy$ by sketching the region R defined by the limits, changing the order of integration to $\int\int_R \sqrt{1 + x^2}\, dy\, dx$ and evaluating.

4. For each of the following series, state whether the series is absolutely convergent, conditionally convergent, or divergent and show why your answer is correct. (No credit for any part unless your reasons are given.)

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{e^{-k}}{\sqrt{k+1}}$

   (b) $\displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{1}{x(\ln(x))^2}$

   (c) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{2k+1}{\sqrt{4k^3+1}}$

5. Find the interval of convergence including possible endpoints for the power series $\sum_{k=2}^{\infty} \frac{x^k}{2^k \ln(k)}$.

6. Suppose the temperature T is given by $T = x^3 + y^3 - 3xy$.

   (a) Find and classify all critical points of T.

   (b) Find the hottest point on the square plate bounded by $x = 0, x = 2$ and $y = 0, y = 3$.

7. Find the volume of the cone with base radius a given by $a - z = \sqrt{x^2 + y^2}$, $z \geq 0$.

Part II: (30 points) Solve 3 complete problems out of 5

8. Let $f(x, y)$ be a differentiable function that is constant on all circles $x^2 + y^2 = a^2$ for any radius $a \geq 0$. If $(x, y) \neq (0, 0)$, show that

$$\frac{f_y(x, y)}{f_x(x, y)} = \frac{y}{x}.$$

   Hint: Observe that in polar coordinates, f is a function of r alone.

9. (a) Find a power series expansion for $\frac{1}{1+t^2}$ and determine its radius of convergence.

   (b) Use the fact that

$$\int_0^{1/\sqrt{3}} \frac{1}{1+t^2} dt = \arctan(t) \Big|_0^{1/\sqrt{3}} = \pi/6$$

   to express $\pi/6$ as an infinite series. What is the fewest number of terms you must calculate to guarantee 2 decimal places of accuracy?

10. Let R be the lamina given by a quarter circle bounded by $x^2 + y^2 = 4$, $x \geq 0$, $y \geq 0$, and suppose the density of the lamina is $\rho = \sqrt{x^2 + y^2}$.

   (a) Find the mass of the lamina.

   (b) Find the the coordinates of the center of mass $(\overline{x}, \overline{y})$ of the lamina.

11. Find the surface area of the part of the sphere of radius 3 given by $x^2 + y^2 + z^2 = 9$ that lies inside the cylinder $x^2 + y^2 = 4$.

12. (a) Evaluate the limit or show that it does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2}.$$

   (b) Expand $f(x) = \frac{1}{x}$ as a Taylor series about $x = 2$. Use the remainder formula to estimate the error in using the first five terms of your series to evaluate $\frac{1}{e}$.