Math 202 Final Exam Spring,2008

Part I Do all parts of the following six problems.

(1) Compute the derivative $\frac{dy}{dx}$ for each of the following (18 points) :

(a) $y = 2^{\arctan(3x)}$;

Ans: $y' = \dfrac{3\ln(2)2^{\arctan(3x)}}{(1+9x^2)}$.

(b) $y = x^2 + x^x$;

Ans: $y' = 2x + x^x(\ln(x)+1)$.

(c) $y = \ln(\sqrt{x^4+3x}\ )$.

Ans: $y' = \dfrac{4x^3+3}{2(x^4+3)}$

(2) Compute each of the following integrals(24 points):

(a) $\displaystyle\int \frac{x^3}{\sqrt{4+x^2}}\, dx$;

Ans: with $u = 4+x^2$ $\displaystyle\int \frac{u-4}{2\sqrt{u}}\, du =$

$\frac{1}{3}(4+x^2)^{3/2} - 4(4+x^2)^{1/2} + C$

Alternatively, use $x = 2\tan(t)$, $\sqrt{4+x^2} = 2\sec(t)$.

(b) $\displaystyle\int \frac{x^3-1}{x^3+x}\, dx$

Ans: $\displaystyle\int 1 + \frac{-1}{x} + \frac{x-1}{x^2+1}\, dx = x - \ln(x) + \frac{1}{2}\ln(x^2+1) - \arctan(x) + C.$

(c) $\displaystyle\int \cos^3(x)\ dx;$

Ans: let $u = \sin(x)$, $\displaystyle\int (1 - u^2)\ du = \sin(x) - \frac{1}{3}\sin^3(x) + C.$

(d) $\displaystyle\int_0^1 x\ln(x+1)\ dx.$

Ans: let $z = x+1, u = \ln(z), dv = (z-1)dz$ $\displaystyle\int_1^2 (z-1)\ln(z)\ dz =$

$[\frac{1}{2}z^2 - z]\ln(z) - [\frac{1}{4}z^2 - z]\ |_1^2\ =\ \frac{1}{4}.$

(3) Compute each of the following limits (10 points):

(a) $Lim_{x\to\infty}\dfrac{x^2 + e^x}{x^3 + e^x};$ Ans $= 1.$

(b) $Lim_{x\to\infty}x^{1/x};$ Ans $= e^0 = 1.$

(4) The region R in first quadrant of the xy plane is bounded by the curves $y = 9 - x^2$, $y = 0$ and $x = 0$. Set up two integrals (method of washers and method of shells) for the volume of the solid obtained by rotating R around the line $x = 10$. Do not compute the value of the integrals(10 points)

$$\text{Ans, washers} = \pi\int_0^9 10^2 - (10 - \sqrt{9-y}\ )^2\ dy;$$

$$\text{shells} = 2\pi\int_0^3 (10-x)(9-x^2\ )\ dx.$$

(5) Sketch the curve given by the equation $r = 2+\cos(\theta)$ in polar coordinates, labeling the x and y intercepts, and compute the area it encloses. (8 points)

$$\text{Area} = \frac{1}{2}\int_0^{2\pi} (2 + cos(\theta))^2\ d\theta = 9\pi.$$

Part II Do all parts of three out of the following four problems (10 points each)

(7) A 10 foot long tank with height 4 feet has a vertical cross-section given by the curve $y = x^2$. It is filled to a height of 3 feet with a liquid having density 30 pounds per cubic foot. Set up an integral for the work necessary to pump the liquid in the tank out over the top. Use the integral to compute the work.

$$\text{Ans} = \int_0^3 (4-y)30 \cdot 10(2\sqrt{y})\, dy = 2640\sqrt{3}.$$

(8) (a) Calculate the arc-length of the section of the curve $y = 2x^{3/2}$ between $x = 0$ and $x = 1$.

$$\text{Ans} = \int_0^1 \sqrt{1 + (3x^{1/2})^2}\, dx = \frac{1}{9}\int_1^{10} u^{1/2}\, du = \frac{1}{9}(10^{3/2} - 1).$$

(b) Evaluate the integral or show it is divergent: $\int_0^\infty te^{-t^2}\, dt$.

$$\text{Ans} = \lim_{M\to\infty} -\frac{1}{2}e^{-t^2/2} \Big|_0^M = \frac{1}{2}.$$

(9) (a) A radioactive substance has a half-life of 64 days. Derive a formula for the amount left after t days if you begin with 500 grams of the substance (Show your work).

$$X = 500\, e^{(\frac{\ln(2)t}{64})}.$$

(b) Find all values of t such that the tangent line to the curve $x = t^3 - t^2 - t, y = \cos(t) + 1$ is horizontal.

$$\frac{dy}{dx} = 0 \quad \text{when} \quad \frac{dy}{dt} = -\sin(t) = 0 \quad \text{so when} \quad t = n\pi \text{ for } n \text{ an integer.}$$

(10) (a) Draw a labeled sketch of the conic whose equation is $9x^2+y^2+2y = 35$. Identify which sort of conic it is.

$$\frac{x^2}{4} + \frac{(y+1)^2}{36} = 1 \quad \text{is an ellipse with}$$

center at $(0, -1)$ and foci at $(0, -1 \pm 4\sqrt{2})$.

(b) Write out the form of the partial fraction decomposition of the following function. Do not attempt to determine the numerical values of the coefficients.

$$\frac{x^2+1}{(x-1)^3(x^2+9)^2}.$$

Ans: $$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+9} + \frac{Fx+G}{(x^2+9)^2}.$$