Department of Mathematics — The City College of New York

Mathematics 34600 — Final Examination — Spring 2006

PART I: Answer any 5 out of 7 questions. Each is worth 16 points.

1. Solve the system of equations.

$$\begin{aligned} x_1 - 2x_2 + x_3 - 4x_4 &= 1 \\ x_1 + 3x_2 + 7x_3 + 2x_4 &= 2 \\ x_1 - 12x_2 - 11x_3 - 16x_4 &= 5 \end{aligned}$$

2. a) Find the inverse of $A = \begin{pmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{pmatrix}$

   b) Use the answer to part a) to solve the system

$$\begin{aligned} 3x_1 + 4x_2 - x_3 &= 1 \\ x_1 + 3x_3 &= 0 \\ 2x_1 + 5x_2 - 4x_3 &= 1 \end{aligned}$$

3. a) Determine whether $W = \left\{ \begin{pmatrix} a & a+2b \\ a-3b & b \end{pmatrix} : a, b \text{ real} \right\}$ is a subspace of $\boldsymbol{M}_{2,2}$.

   b) Find a basis for the subspace of $\boldsymbol{R}^4$ spanned by the vectors $v_1 = (1,1,0,0)$, $v_2 = (0,0,1,1)$, $v_3 = (-2,0,2,2)$, $v_4 = (0,-3,0,3)$.

4. Let $A = \begin{pmatrix} 3 & -1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & -3 & 0 & 1 \end{pmatrix}$

   a) Find a basis for the nullspace of A.
   b) Find a basis for the column space of A.
   c) Compute Nullity (A^T).

5. Determine whether the following linear operator is one-to-one and if possible compute $T^{-1}(w_1, w_2, w_3)$.

$$\begin{aligned} w_1 &= 2x_1 + 2x_2 + x_3 \\ w_2 &= 2x_1 + x_2 - x_3 \\ w_3 &= 3x_1 + 2x_2 + x_3 \end{aligned}$$

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6. a) Find a basis for the subspace $W = \{(x, y, z) : 3z = 7y\}$ of $\boldsymbol{R}^3$.
   b) Determine the dimension of the following subspace of $\boldsymbol{R}^4$
   $W = \{(a, b, c, d) : a = b + c,\ \ b + 2d - c = 0\}$
   c) Find the coordinate vector of $w = (a, b)$ relative to the basis $S = \{u_1, u_2\}$ of $\boldsymbol{R}^2$ where $u_1 = (1,2)$ and $u_2 = (-1,3)$.

7. a) A square matrix is called symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$.
   Show that if B is a square matrix, then
   (i) BB^T and $B + B^T$ are symmetric.
   (ii) $B - B^T$ is skew-symmetric.
   b) Show that if $\{v_1, v_2, v_3\}$ is a linearly dependent set of vectors in a vector space V, and v_4 is any vector in V, then $\{v_1, v_2, v_3, v_4\}$ is also linearly dependent.

<u>PART II</u>: Answer 1 of the 2 questions (20 points).

8. Let $A = \begin{pmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{pmatrix}$.
   a) Diagonalize A.
   b) Use the result from part a) to compute A^{10}.

9. Let W be the subspace of $\boldsymbol{P_2}$ spanned by $S = \{p_1, p_2, p_3, p_4\}$ where $p_1 = 1 + 3x + 3x^2$ $p_2 = x + 4x^2$, $p_3 = 5 + 6x + 3x^2$, $p_4 = 7 + 2x - x^2$.
   a) Find a subset of S which is a basis for W.
   b) Express the element(s) of S not in the basis you found in part a) as linear combinations of the basis elements.
   c) (Independent from parts a) and b)) Determine whether $T(x, y) = (x, 2y - 3)$ is a linear operator.