PART I: Answer any 5 out of 7 questions. Each is worth 16 points.

1. Solve the system of equations.
\[ \begin{align*}
    x_1 - 2x_2 + x_3 - 4x_4 &= 1 \\
    x_1 + 3x_2 + 7x_3 + 2x_4 &= 2 \\
    x_1 - 12x_2 - 11x_3 - 16x_4 &= 5
\end{align*} \]

2. a) Find the inverse of
\[
\begin{pmatrix}
3 & 4 & -1 \\
1 & 0 & 3 \\
2 & 5 & -4
\end{pmatrix}
\]
b) Use the answer to part a) to solve the system
\[ \begin{align*}
    3x_1 + 4x_2 - x_3 &= 1 \\
    x_1 + 3x_3 &= 0 \\
    2x_1 + 5x_2 - 4x_3 &= 1
\end{align*} \]

3. a) Determine whether \( W = \left\{ \begin{pmatrix} a & a + 2b \\ a - 3b & b \end{pmatrix} : a, b \text{ real} \right\} \) is a subspace of \( \mathbb{M}_{2,2} \).
b) Find a basis for the subspace of \( \mathbb{R}^4 \) spanned by the vectors \( \mathbf{v}_1 = (1, 1, 0, 0) \), \( \mathbf{v}_2 = (0, 0, 1, 1) \), \( \mathbf{v}_3 = (-2, 0, 2, 2) \), \( \mathbf{v}_4 = (0, -3, 0, 3) \).

4. Let \( A = \begin{pmatrix}
3 & -1 & 2 & 1 \\
2 & 1 & 1 & 1 \\
1 & -3 & 0 & 1
\end{pmatrix} \)
a) Find a basis for the nullspace of \( A \).
b) Find a basis for the column space of \( A \).
c) Compute Nullity \( (A^T) \).

5. Determine whether the following linear operator is one-to-one and if possible compute \( T^{-1}(w_1, w_2, w_3) \).
\[ \begin{align*}
    w_1 &= 2x_1 + 2x_2 + x_3 \\
    w_2 &= 2x_1 + x_2 - x_3 \\
    w_3 &= 3x_1 + 2x_2 + x_3
\end{align*} \]

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6. a) Find a basis for the subspace \( W = \{(x, y, z) : 3z = 7y\} \) of \( \mathbb{R}^3 \).

b) Determine the dimension of the following subspace of \( \mathbb{R}^4 \)
\[ W = \{(a, b, c, d) : a = b + c, \ b + 2d - c = 0\} \]
c) Find the coordinate vector of \( w = (a, b) \) relative to the basis \( S = \{u_1, u_2\} \) of \( \mathbb{R}^2 \) where \( u_1 = (1,2) \) and \( u_2 = (-1,3) \).

7. a) A square matrix is called symmetric if \( A^T = A \) and skew-symmetric if \( A^T = -A \).

Show that if \( B \) is a square matrix, then
(i) \( BB^T \) and \( B + B^T \) are symmetric.
(ii) \( B - B^T \) is skew-symmetric.

b) Show that if \( \{v_1, v_2, v_3\} \) is a linearly dependent set of vectors in a vector space \( V \), and \( v_4 \) is any vector in \( V \), then \( \{v_1, v_2, v_3, v_4\} \) is also linearly dependent.

**PART II:** Answer 1 of the 2 questions (20 points).

8. Let \( A = \begin{pmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{pmatrix} \).

a) Diagonalize \( A \).

b) Use the result from part a) to compute \( A^{10} \).

9. Let \( W \) be the subspace of \( \mathbb{P}_2 \) spanned by \( S = \{p_1, p_2, p_3, p_4\} \) where
\[ p_1 = 1 + 3x + 3x^2 \]
\[ p_2 = x + 4x^2, \quad p_3 = 5 + 6x + 3x^2, \quad p_4 = 7 + 2x - x^2. \]

a) Find a subset of \( S \) which is a basis for \( W \).

b) Express the element(s) of \( S \) not in the basis you found in part a) as linear combinations of the basis elements.

c) (Independent from parts a) and b)) Determine whether \( T(x, y) = (x, 2y - 3) \) is a linear operator.