PART I: ANSWER ALL THREE QUESTIONS (40%)

1. (15) Let \( A = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \).

   a. Find the eigenvalues and bases for the eigenspaces of \( A \).
   b. Find a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( P^{-1}AP = D \).
   c. Use the result of part b. to compute \( A^5 \).

2. (10) Let \( T: V \rightarrow W \) be a linear transformation.
   a. Define the kernel of \( T \) and prove that it is subspace of \( V \).
   b. Define the range of \( T \) and prove that it is subspace of \( W \).

3. (15) Suppose \( L: P_2 \rightarrow P_2 \) is given by \( L(p(t)) = 2p(t) - 3p'(t) \). Given the basis \( B = \{1, 3t + 1, 2t^2\} \) for \( P_2 \),
   a. Find the \( B \)-coordinates of \( p(t) = 5 - 6t + 4t^2 \).
   b. Find \( [L]_B \), the \( B \)-matrix for \( L \).
   c. Using the results of parts a. and b., compute \( [L(5 - 6t + 4t^2)]_B \).

PART II: ANSWER FIVE OUT OF SEVEN QUESTIONS (OMIT TWO) (60%)

4. (12) Suppose that \( S = \{v_1, v_2\} \) is a linear independent set and that \( v \not\in \text{Span}(S) \).
   Prove that \( \{v, v_1, v_2\} \) is a linearly independent set.

5. (12) Let \( A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \).
   a. Find a basis for the nullspace of \( A \).
   b. Find a basis for the column space of \( A \).
   c. What is the rank of \( A \)?

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6. (12) a. Find $A^{-1}$ if $A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & 5 & 2 & 2 \\ 1 & 3 & 8 & 9 \\ 1 & 3 & 2 & 2 \end{bmatrix}$.

b. Solve $Ax = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ using your answer from part a.

7. (12) Let $B$ be the reduced row echelon form of an $m \times n$ matrix $A$. For each of the following statements, if it is true, explain why. If it is false, give a counterexample.

a. Row space of $A = \text{Row space of } B$

b. Column space of $A = \text{Column space of } B$

8. (12) Use Cramer's Rule (no credit for any other method) to solve the system

\[
\begin{align*}
4x_1 + 5x_2 &= 2 \\
11x_1 + x_2 + 2x_3 &= 3 \\
x_1 + 5x_2 + 2x_3 &= 1
\end{align*}
\]

9. (12) Suppose $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation. Explain your answers to the following questions:

a. Under what conditions on $m$ and $n$ can one be sure that $T$ is not one-to-one?

b. Under what conditions on $m$ and $n$ can one be sure that $T$ is not onto?

c. If $T$ is one-to-one and onto, what can one conclude about $m$ and $n$?

10. (12) Let $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$.

a. Find $AB$ and $BA$.

b. Show that $\{A, B, AB, BA\}$ is a basis for $M_{2 \times 2}$, the vector space of all $2 \times 2$ real matrices.