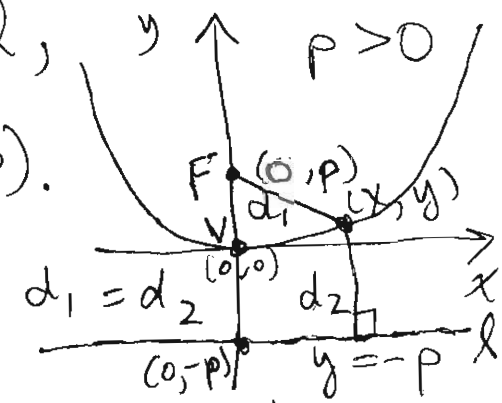
# Conics

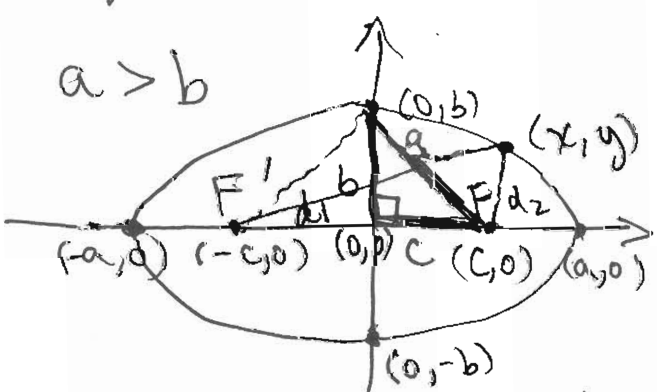


<u>Parabola</u> is a curve whose points are equi-distant from a fixed point called the <u>focus</u> F and a fixed line (that does not contain F) called the <u>directrix</u> $l$, a distance $2p$ apart. The <u>vertex</u> V of the parabola is halfway between F & $l$ ($d_1 = d_2 = p$). In the standard orientation to the right, $x^2 = 4py$; $x^2 = -4py$ is the reflection in the x-axis, while $y^2 = \pm 4px$ are the reflections in the line $y = x$ and open to the right and left, respectively.




<u>Ellipse</u> is a curve whose points have the property that their distances from two fixed points F′, F called the <u>foci</u> (plural of focus) sums to a constant $= 2a$ when

a > b



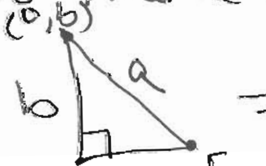

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ in standard form.}$$

$$y = 0 \longrightarrow x = \pm a$$
$$x = 0 \longrightarrow y = \pm b$$

$d_1 + d_2 = \text{const.} = 2a$ applied to $(0, b)$

$\Longrightarrow d_1 = d_2 = a$ due to symmetry

so there is a right triangle (see sketch)

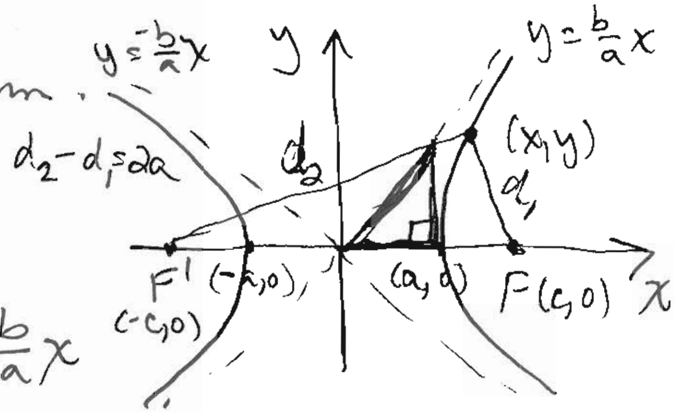



$\Longrightarrow a^2 = b^2 + c^2$. The <u>intercepts</u> are called the <u>vertices</u>, the longer line between the vertices also carries the foci and is called the <u>major axis</u>, the line segment joining the other pair of vertices is the <u>minor axis</u>. If $b > a$ in the standard equation, the roles of $x$ + $y$ are interchanged and the major axis then lies along the $y$-axis. Either way, the <u>center</u> of the ellipse is the origin $(0,0)$ and

$$\left(\tfrac{1}{2}\text{ length of major axis}\right)^2 = \left(\tfrac{1}{2}\text{ length minor axis}\right)^2 + (\text{dist. bet. } (0,0) \text{ \& } F)^2.$$

<u>Hyperbola</u> is a curve whose points have the property that the difference of their distances from two fixed points $F'$, $F$ (also called the <u>foci</u>) is a constant $= 2a$ when

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ in standard form.}$$

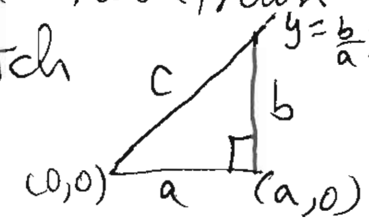



$x = 0$ has no solution in $y$

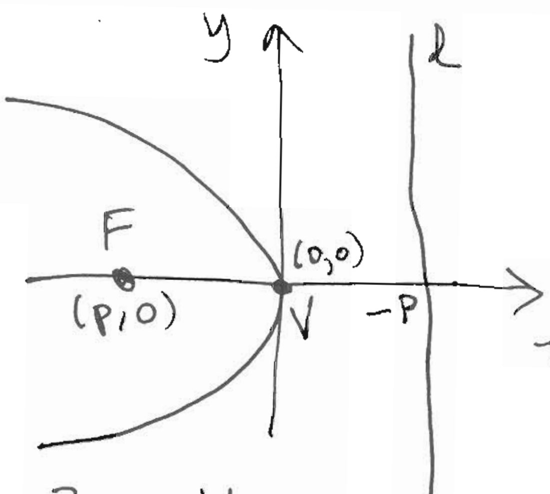
$y = 0 \Rightarrow x = \pm a$

$$\lim_{x \to \pm\infty} y = \lim_{x \to \pm\infty} \pm\sqrt{b^2\left(\frac{x^2}{a^2} - 1\right)} = \pm\frac{b}{a}x$$

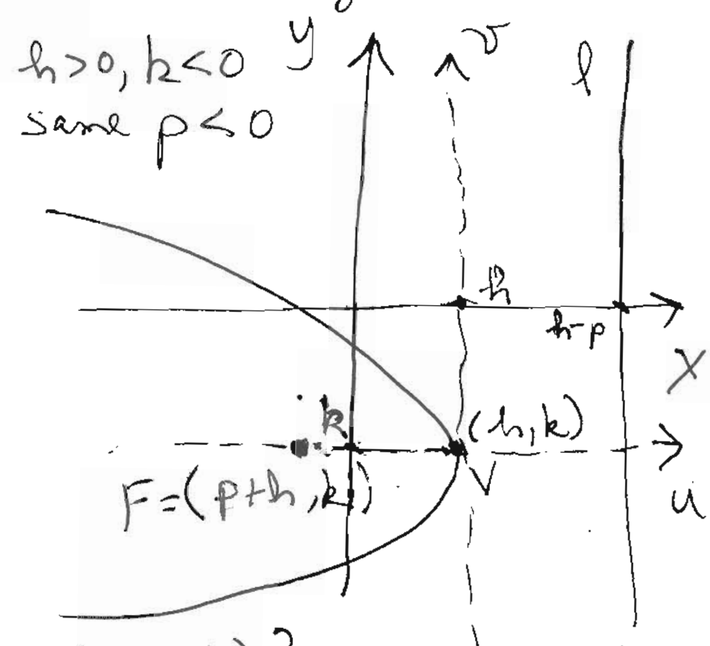
so the lines $y = \pm\frac{b}{a}x$ are <u>oblique</u> (or <u>slant</u>) <u>asymptotes</u> (see Stewart 3.4 # 39–42). Since slope = rise/run the right triangle formed in the sketch has the property $c^2 = a^2 + b^2$:



Translation is done just as in calc. I: if $f(x,y) = \text{const.}$ represents a curve, then $f(x-h, y-k) = \text{same const.}$ represents the same geometric curve translated to a new rectangular coordinate system whose origin is $(h,k)$ and whose axes are $x = h$ and $y = k$, i.e. letting $u = x - h$ and $v = y - k$ one obtains $f(x,y) = f(u,v)$; $f(0,0) = f(h,k)$ and the curve relative to the $uv$-coordinate system looks like the original curve relative to the $xy$-coordinates.



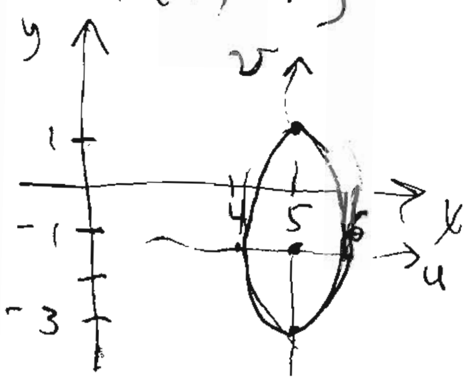

$y^2 = 4px$, $p < 0$



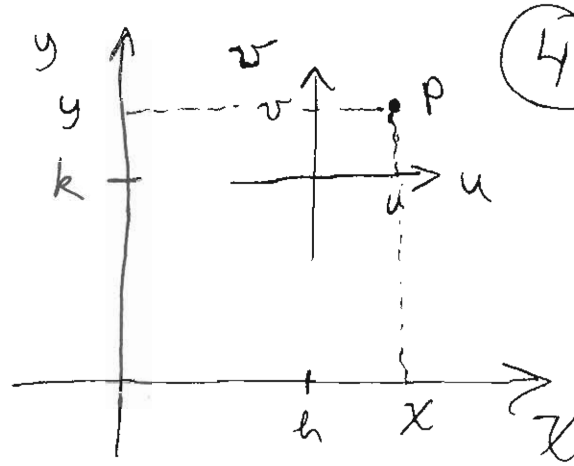

$(y-k)^2 = 4p(x-h)$

So, for example, an ellipse whose vertices are $(5,-3)$, $(5,1)$, $(4,-1)$ and $(6,-1)$ is a vertical ellipse whose center is $(5,-1)$ with

$$\frac{u^2}{1} + \frac{v^2}{2^2} = 1 = (x-5)^2 + \frac{(y+1)^2}{2^2}$$



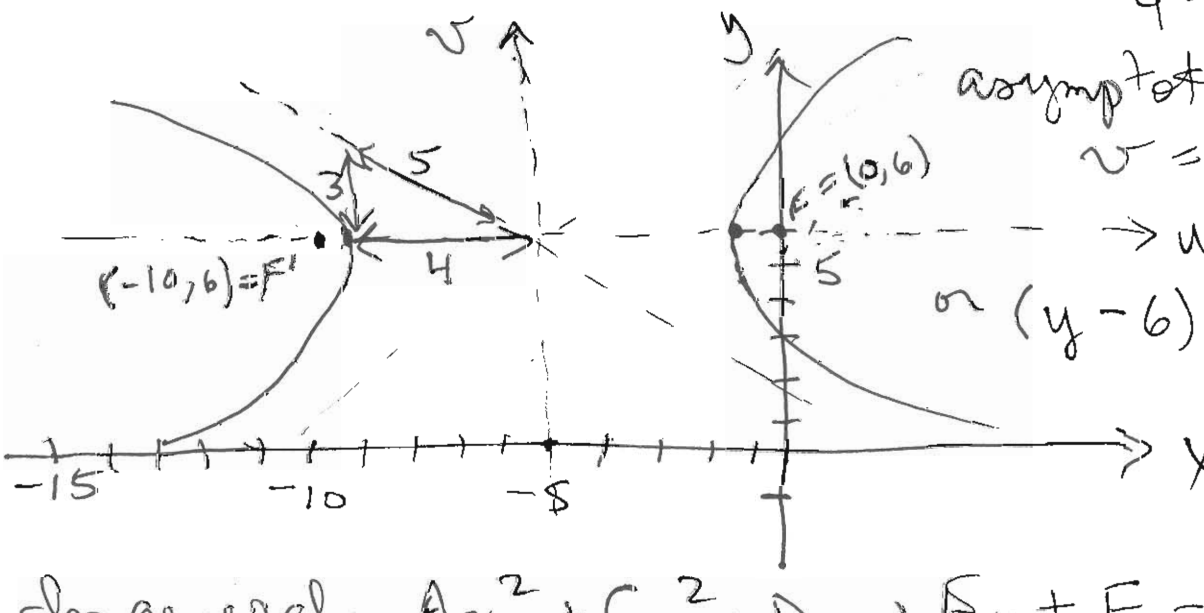
because a point P whose coordinates are $(u, v)$ in the translated system has $(k+v)$ as its y-coordinate and $(h+u)$ as its x-coordinate:




Alternatively, given $9x^2-16y^2+90x+192y=495$ completing the squares in $x$ and $y$ gives

$$\frac{(x+5)^2}{16}-\frac{(y-6)^2}{9}=1$$

which is a horizontal (no $v$-intercepts possible) hyperbola with center $(-5, 6)=(h, k)$, $(-5\pm\sqrt{16}, 6)$ as vertices, i.e. $(-9,6)$ + $(-1,6)$, and foci $(-5\pm\sqrt{25}, 6)$, since $\frac{u^2}{4^2}-\frac{v^2}{3^2}=1$




asymptotes are $v=\pm\frac{3}{4}u$

or $(y-6)=\pm\frac{3}{4}(x+5)$

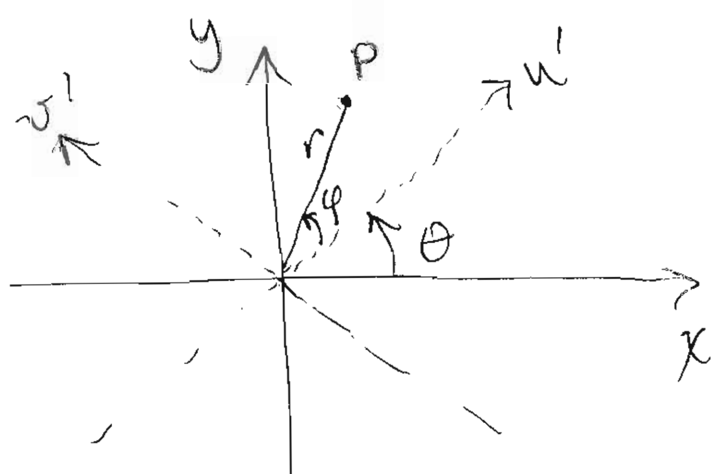
In general, $Ax^2+Cy^2+Dx+Ey+F=0$ is a translated conic, a degenerate conic (point, line, pair of lines, eg. when $x^2+y^2=0$, $x=$ const, $y^2=$ const giving ∥ lines or $x^2-y^2=0$ gives an intersecting pair), or has no solution (eg. $x^2+y^2=\text{const.}<0$).

<u>Rotation</u> occurs when the cross-term in the most general quadratic in x and y is non-zero: $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$.

To eliminate the cross-term, the Cartesian coordinate system (i.e. the xy or uv-coordinate system, depending on whether translation is also a factor) must be rotated by $0 \leq \theta \leq \frac{\pi}{2}$ in the counterclockwise direction, where

$$\cot 2\theta = \frac{A-C}{B}$$

: for example, in the absence of translation if the point P is a distance r from the mutual origin of the xy and u'v'-coordinate systems, and the




radial line from (0,0) to P makes an angle $\varphi$ with respect to the u'-axis, then it makes an angle of $(\varphi+\theta)$ with respect to the x-axis, etc., and $P = (\overbrace{r\cos\varphi}^{u'}, \overbrace{r\sin\varphi}^{v'})$ in the u'v'-coordinate system, but in the xy-system P has coordinates $(r\cos(\theta+\varphi), r\sin(\theta+\varphi))$

$$= (\underbrace{u'\cos\theta - v'\sin\theta}_{x}, \underbrace{u'\sin\theta + v'\cos\theta}_{y}).$$