

When we want to find the first derivative when both x and y are dependent functions, we must use the following formula:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

For the second derivative

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

Therefore, for the higher derivatives:

$$\frac{d^n y}{dx^n} = \frac{\frac{d}{dt}\left(\frac{d^{(n-1)}y}{dx^{(n-1)}}\right)}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

The arc length formula is located on previous page.

If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous on the interval $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β , then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Additional examples:

$$2) \quad x = \frac{1}{t} \quad y = \sqrt{t}e^{-t}$$

$$\frac{dx}{dt} = \frac{-1}{t^2} \quad \frac{dy}{dt} = \left[\frac{1}{2\sqrt{t}} \right] (e^{-t}) + (\sqrt{t})[-e^{-t}] = \frac{e^{-t}}{2\sqrt{t}} - \sqrt{t}e^{-t} = \frac{e^{-t} - 2te^{-t}}{2\sqrt{t}}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\left(\frac{e^{-t} - 2te^{-t}}{2\sqrt{t}} \right)}{\left(\frac{-1}{t^2} \right)} = \left(\frac{e^{-t} - 2te^{-t}}{2\sqrt{t}} \right) \left(\frac{-t^2}{1} \right) = \frac{2t^2\sqrt{t}e^{-t} - t\sqrt{t}e^{-t}}{2}$$

$$6) \quad x = \sin^3 \theta \quad y = \cos^3 \theta \quad \theta = \frac{\pi}{6}$$

$$\frac{dx}{d\theta} = 3\sin^2 \theta \cos \theta \quad \frac{dy}{d\theta} = 3\cos^2 \theta (-\sin \theta) = -3\sin \theta \cos^2 \theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3\sin \theta \cos^2 \theta}{3\sin^2 \theta \cos \theta} = \frac{-\cos \theta}{\sin \theta}$$

$$m = \frac{-\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)} = \frac{-\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = -\sqrt{3} \quad x = \sin^3\left(\frac{\pi}{6}\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \quad y = \cos^3\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$$

$$y - \frac{3\sqrt{3}}{8} = -\sqrt{3}\left(x - \frac{1}{8}\right) \Rightarrow y = -\sqrt{3}x + \frac{\sqrt{3}}{2}$$

$$10) \quad x = t^3 + 1 \quad y = t^2 - t$$

$$\frac{dx}{dt} = 3t^2 \quad \frac{dy}{dt} = 2t - 1 \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 1}{3t^2}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{2t - 1}{3t^2}\right) = \frac{[2](3t^2) - (2t - 1)[6t]}{(3t^2)^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\left(\frac{6t - 6t^2}{(3t^2)^2}\right)}{(3t^2)} = \frac{6t - 6t^2}{(3t^2)^3}$$

$$14) \quad x = t^3 - 3t \quad y = t^3 - 3t^2$$

$$\frac{dx}{dt} = 3t^2 - 3 \quad \frac{dy}{dt} = 3t^2 - 6t$$

$$0 = 3t^2 - 6t$$

$$0 = 3t^2 - 3$$

$$\text{Horizontal: } \frac{dy}{dt} = 0 \quad 0 = 3t(t-2) \\ 3t = 0 \quad t-2 = 0 \\ t = 0 \quad t = 2$$

$$\text{Vertical: } \frac{dx}{dt} = 0 \quad 0 = 3(t+1)(t-1) \\ t+1 = 0 \quad t-1 = 0 \\ t = -1 \quad t = 1$$

16) $x = e^{\sin \theta} \quad y = e^{\cos \theta}$

$$\frac{dx}{d\theta} = e^{\sin \theta}(\cos \theta) = \cos \theta e^{\sin \theta} \quad \frac{dy}{d\theta} = e^{\cos \theta}(-\sin \theta) = -\sin \theta e^{\cos \theta} \quad \text{note: } e^x \geq 0 \text{ for all } x.$$

Horizontal: $\frac{dy}{d\theta} = 0$ Vertical: $\frac{dx}{d\theta} = 0$

$$0 = -\sin \theta e^{\cos \theta}$$

$$0 = \cos \theta e^{\sin \theta}$$

$$\cos \theta = 0$$

$$\sin \theta = 0$$

$$\theta = \frac{\pi}{2}$$

$$\theta = \frac{3\pi}{2}$$

$$\theta = k\pi \quad \text{where } k \text{ is any integer}$$

$$\theta = 0 \quad \theta = \pi$$

$$\theta = 0 + 2k\pi \quad \theta = \pi + 2k\pi$$

$$\theta = \frac{\pi}{2} + 2k\pi \quad \theta = \frac{3\pi}{2} + 2k\pi$$

36) $x = t + \sqrt{t} \quad y = t - \sqrt{t} \quad 0 \leq t \leq 1$

$$\frac{dx}{dt} = 1 + \frac{1}{2\sqrt{t}} \quad \frac{dy}{dt} = 1 - \frac{1}{2\sqrt{t}} \quad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{\left(1 + \frac{1}{2\sqrt{t}}\right)^2 + \left(1 - \frac{1}{2\sqrt{t}}\right)^2} \quad L = \int_0^1 \sqrt{\left(1 + \frac{1}{2\sqrt{t}}\right)^2 + \left(1 - \frac{1}{2\sqrt{t}}\right)^2} dt$$

38) $x = e^t + e^{-t} \quad y = 5 - 2t \quad 0 \leq t \leq 3$

$$\frac{dx}{dt} = e^t - e^{-t} \quad \frac{dy}{dt} = -2 \quad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(e^t - e^{-t})^2 + (-2)^2} = \sqrt{(e^{2t} - 2 + e^{-2t}) + 4} = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = (e^t + e^{-t})$$

$$L = \int_0^3 (e^t + e^{-t}) dt = \left[e^t - e^{-t} + C \right]_0^3 = \left[e^{(3)} - e^{-(3)} + C \right] - \left[e^{(0)} - e^{-(0)} + C \right] = \left[e^3 - \frac{1}{e^3} \right] - [1 - 1] = e^3 - \frac{1}{e^3}$$

40) $x = 3 \cos t - \cos 3t \quad y = 3 \sin t - \sin 3t \quad 0 \leq t \leq \pi$

$$\frac{dx}{dt} = -3 \sin t + 3 \sin 3t \quad \frac{dy}{dt} = 3 \cos t - 3 \cos 3t \quad L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= 3(-\sin t + \sin 3t) \quad = 3(\cos t - \cos 3t)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$$

$$2 \sin^2 \theta = (1 - \cos(2\theta))$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3(-\sin t + \sin 3t))^2 + (3(\cos t - \cos 3t))^2}$$

$$= \sqrt{9(\sin^2 t - 2 \sin t \sin 3t + \sin^2 3t) + 9(\cos^2 t - 2 \cos t \cos 3t + \cos^2 3t)}$$

$$= 3\sqrt{\sin^2 t + \cos^2 t + \sin^2 3t + \cos^2 3t - 2 \sin t \sin 3t - 2 \cos t \cos 3t}$$

$$= 3\sqrt{1 + 1 - 2(\cos 3t \cos t + \sin 3t \sin t)} = 3\sqrt{2 - 2(\cos(3t - t))}$$

$$= 3\sqrt{2(1 - \cos(2t))} = 3\sqrt{2(2 \sin^2 t)} = 6 \sin t$$

$$L = \int_0^\pi 6 \sin t dt = \left[-6 \cos t + C \right]_0^\pi = \left[-6 \cos(\pi) + C \right] - \left[-6 \cos(0) + C \right] = \left[-6(-1) \right] - \left[-6(1) \right] = 12$$