We are covering up to example 4.

One of the formulas needed for this section is Hooke's Law which states that the force required to maintain a spring stretched or compressed *x* units beyond its natural length is proportional to *x*: $f(x) = kx$
where *k* is positive and called spring constant.

To be successful in this section we must know the measurement types of both US customary (imperial) and SI (MKS) measurement system.

Measurement type	**US customary**	**SI**
Length	Foot (ft)	Meter (m)
Mass	Slug {mostly, not used in our examples}	Kilogram (Kg)
Force	Pounds (lbs)	Newton ($\mathrm{N} = (\mathrm{Kg})(\mathrm{m/sec}^2)$)
Work	Foot-pound (ft-lbs)	Joules ($\mathrm{J} = (\mathrm{N})(\mathrm{m})$)

For the rest of the material, use the handout given with syllabus (or notes for Chapter 7).

Additional Examples:

2) $F(x) = \cos\left(\frac{\pi}{3}x\right) \qquad W = \int_a^b F(x)\,dx = \int_a^b \cos\left(\frac{\pi}{3}x\right)dx = \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}x\right) + C\right]_a^b$

$$W_{1-2} = \int_1^2 \cos\left(\frac{\pi}{3}x\right)dx = \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}x\right) + C\right]_1^2 = \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}(2)\right) + C\right] - \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}(1)\right) + C\right]$$

$$= \left[\frac{3}{\pi}\sin\left(\frac{2\pi}{3}\right)\right] - \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}\right)\right] = \left[\frac{3}{\pi}\left(\frac{\sqrt{3}}{2}\right)\right] - \left[\frac{3}{\pi}\left(\frac{\sqrt{3}}{2}\right)\right] = 0 \text{ J}$$

$$W_{1-1.5} = \int_1^{1.5} \cos\left(\frac{\pi}{3}x\right)dx = \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}x\right) + C\right]_1^{1.5} = \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}(1.5)\right) + C\right] - \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}(1)\right) + C\right]$$

$$= \left[\frac{3}{\pi}\sin\left(\frac{\pi}{2}\right)\right] - \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}\right)\right] = \left[\frac{3}{\pi}(1)\right] - \left[\frac{3}{\pi}\left(\frac{\sqrt{3}}{2}\right)\right] = \frac{3}{\pi}\left(1 - \frac{\sqrt{3}}{2}\right) \text{ J}$$

$$W_{1.5-2} = \int_{1.5}^2 \cos\left(\frac{\pi}{3}x\right)dx = \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}x\right) + C\right]_{1.5}^2 = \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}(2)\right) + C\right] - \left[\frac{3}{\pi}\sin\left(\frac{\pi}{3}(1.5)\right) + C\right]$$

$$= \left[\frac{3}{\pi}\sin\left(\frac{2\pi}{3}\right)\right] - \left[\frac{3}{\pi}\sin\left(\frac{\pi}{2}\right)\right] = \left[\frac{3}{\pi}\left(\frac{\sqrt{3}}{2}\right)\right] - \left[\frac{3}{\pi}(1)\right] = \frac{3}{\pi}\left(\frac{\sqrt{3}}{2} - 1\right) \text{ J}$$

The work from 1.5 to 2 is equal in magnitude than 1 to 1.5 but in opposite direction.

8) let L be the natural length of the spring in meters. $F(x)=kx$ $\int kx\,dx=\frac{k}{2}x^2+C$

	Start length	Start displacement	End length	End displacement	Work
1	10 cm	$(0.10-L)$ m	12 cm	$(0.12-L)$ m	6 J
2	12 cm	$(0.12-L)$ m	14 cm	$(0.14-L)$ m	10 J

$$W_1=\int_{(0.10-L)}^{(0.12-L)} kx\,dx=\left[\frac{k}{2}x^2+C\right]_{(0.10-L)}^{(0.12-L)}=\left[\frac{k}{2}(0.12-L)^2+C\right]-\left[\frac{k}{2}(0.10-L)^2+C\right]$$

$$=\frac{k}{2}\left\{\left[0.0144-0.24L-L^2\right]-\left[0.01-0.2L-L^2\right]\right\}=\frac{k}{2}\left\{0.0044-0.04L\right\}$$

$$W_1=6=\frac{k}{2}\left\{0.0044-0.04L\right\} \qquad 12=0.0001k\left\{44-400L\right\}$$

$$12=k\left\{0.0044-0.04L\right\} \quad\Rightarrow\quad 120000=k\left\{44-400L\right\}$$

$$W_2=\int_{(0.12-L)}^{(0.14-L)} kx\,dx=\left[\frac{k}{2}x^2+C\right]_{(0.12-L)}^{(0.14-L)}=\left[\frac{k}{2}(0.14-L)^2+C\right]-\left[\frac{k}{2}(0.12-L)^2+C\right]$$

$$=\frac{k}{2}\left\{\left[0.0196-0.28L-L^2\right]-\left[0.0144-0.24L-L^2\right]\right\}=\frac{k}{2}\left\{0.0052-0.04L\right\}$$

$$W_2=10=\frac{k}{2}\left\{0.0052-0.04L\right\} \qquad 20=0.0001k\left\{52-400L\right\}$$

$$20=k\left\{0.0052-0.04L\right\} \quad\Rightarrow\quad 200000=k\left\{52-400L\right\}$$

Now we have 2 equations with 2 unknowns. Since we want L we solve W_1 and get $k=\dfrac{120000}{44-400L}$.

$$200000=\left(\frac{120000}{44-400L}\right)\left\{52-400L\right\}$$

$$\frac{200000}{120000}=\frac{52-400L}{44-400L}$$

$$\frac{5}{3}=\frac{52-400L}{44-400L}$$

$$\Rightarrow\quad 5(44-400L)=3(52-400L), \quad 220-2000L=156-1200L, \quad 64=800L$$

$$\Rightarrow\quad L=\frac{64}{800}=\frac{8}{100}\text{ m}=0.08\text{ m}=80\text{ cm}$$

12) Since the weight of the bucket does not change, $W_B = (4 \text{ lbs/ft})(80 \text{ ft}) = 320 \text{ ft-lbs}$.

We are loosing water at a constant rate and the bucket is moving at a constant speed but we need the rate of water lost per distance traveled in order for us to compute our work.

The bucket is being pulled up at a rate of $2 \text{ ft/sec} = \dfrac{2 \text{ ft}}{1 \sec}$ and the water leaking is $0.2 \text{ lbs/sec} = \dfrac{0.2 \text{ lbs}}{1 \sec}$.

Therefore, the rate of water lost over the distance is $\dfrac{\frac{0.2 \text{ lbs}}{1 \sec}}{\frac{2 \text{ ft}}{1 \sec}} = 0.1 \text{ lbs/ft} = \dfrac{1}{10} \text{ lbs/ft}$ and for any arbitrary distance (x ft) traveled be the bucket, we lost $\left(\dfrac{1}{10} \text{ lbs/ft}\right)(x \text{ ft}) = \dfrac{1}{10} x \text{ lbs}$.

We started with 40 lbs of water and at a height x ft, we have $\left(40 - \dfrac{1}{10}x\right) \text{lbs}$ of water left in the bucket.

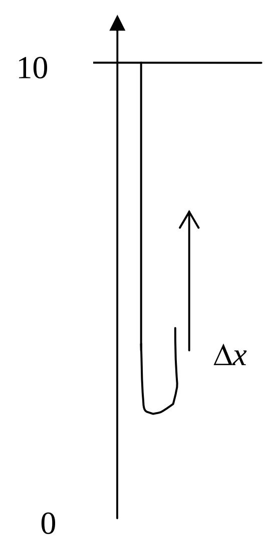
$$W_{water} = \int_0^{80} \left(40 - \frac{1}{10}x\right) dx = \left[40x - \frac{1}{10}x^2 + C\right]_0^{80} = \left[40(80) - \frac{1}{10}(80)^2 + C\right] - \left[40(0) - \frac{1}{10}(0)^2 + C\right]$$

$$= \left[40(80) - \frac{1}{10}(80)(80)\right] - [0] = (3200 - 320) \text{ ft-lbs}$$

$$W = W_B + W_{water} = (320) + (3200 - 320) = 3200 \text{ ft-lbs}$$

14) The rate of chain weight per foot is $\dfrac{25 \text{ lbs}}{10 \text{ ft}} = 2.5 \text{ lbs/ft}$. Let's analyze by link and Δx be the length of the link. The lowest link of chain must travel $10 \text{ ft} = 10 - 0(\Delta x) \text{ ft}$. The next link does not reach the top because only the lowest link occupies this space. So, 2nd link travels $(10 - 2(\Delta x)) \text{ ft}$ and 3rd link travels $(10 - 2(2\Delta x)) = (10 - 4(\Delta x)) \text{ ft}$. This continues until we fold the chain which is 5 ft. Therefore, our interval is $0 \le x \le 5$ and we can formulate the amount of chain that will affect the force:

$F(x) = (2.5 \text{ lbs/ft})((10 - 2x) \text{ ft}) = (25 - 5x) \text{ lbs}$ for any $0 \le x \le 5$



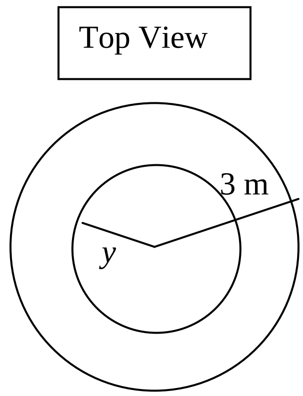
$$W = \int_0^5 (25 - 5x)\, dx = \left[25x - \frac{5}{2}x^2 + C\right]_0^5$$

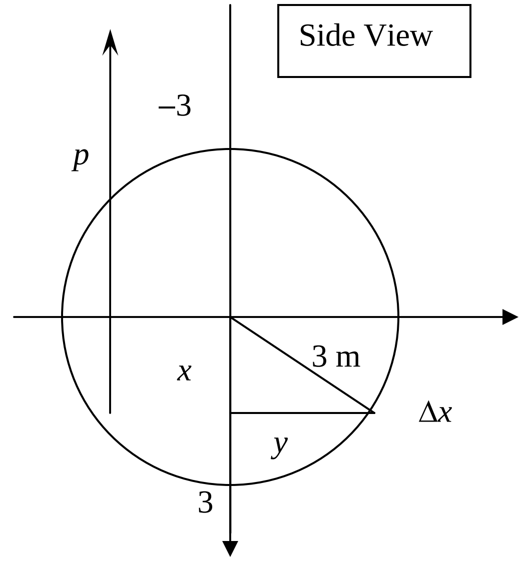
$$= \left[25(5) - \frac{5}{2}(5)^2 + C\right] - \left[25(0) - \frac{5}{2}(0)^2 + C\right]$$

$$= 125\left[1 - \frac{1}{2}\right] = \frac{125}{2} \text{ ft-lbs}$$

18) pumping distance: $p=(x+4)$

$$x^2+y^2=(3)^2$$
$$y^2=9-x^2$$
$$y=\pm\sqrt{9-x^2}$$
$$y=\sqrt{9-x^2}$$
$$A=\pi y^2=\pi\left(\sqrt{9-x^2}\right)^2$$
$$=\pi(9-x^2)\text{ m}^2$$
$$\Delta V=A\Delta x$$
$$=\pi(9-x^2)\Delta x\text{ m}^3$$





a) There is an error in this part, the density here should be $\delta_{water}=1000\text{ kg/m}^3$ because the measurement of the tank is metric. The interval is $-3\le x\le 3$.

$$\Delta m=(\delta)(\Delta V)=(1000\text{ kg/m}^3)(\pi(9-x^2)\Delta x\text{ m}^3)=1000\pi(9-x^2)\Delta x\text{ kg}$$
$$\Delta F=(\Delta m)(a)=(1000\pi(9-x^2)\Delta x\text{ kg})(9.8\text{ m/sec}^2)=9800\pi(9-x^2)\Delta x\text{ N}$$
$$\Delta W=(\Delta F)(p)=(9800\pi(9-x^2)\Delta x\text{ N})((x+4)\text{ m})=9800\pi(36+9x-4x^2-x^3)\Delta x\text{ J}$$
$$W=\int_{-3}^{3}9800\pi(36+9x-4x^2-x^3)\,dx=9800\pi\left[36x+\frac{9}{2}x^2-\frac{4}{3}x^3-\frac{1}{4}x^4+C\right]_{-3}^{3}$$
$$=9800\pi\left\{\left[36(3)+\frac{9}{2}(3)^2-\frac{4}{3}(3)^3-\frac{1}{4}(3)^4+C\right]-\left[36(-3)+\frac{9}{2}(-3)^2-\frac{4}{3}(-3)^3-\frac{1}{4}(-3)^4+C\right]\right\}$$
$$=9800\pi\left\{3^2\left[12(1)+\frac{9}{2}-4-\frac{9}{4}\right]-\left[12(-1)+\frac{9}{2}+4-\frac{9}{4}\right]\right\}=9800\pi\left\{3^2\left[8+\frac{9}{2}-\frac{9}{4}\right]-\left[-8+\frac{9}{2}-\frac{9}{4}\right]\right\}$$
$$=9800\pi(9)(16)=1411200\pi\text{ J}$$

b) The oil has density of $\delta_{oil}=900\text{ kg/m}^3$ and the tank is only half full. The interval is $0\le x\le 3$.

$$\Delta m=(\delta)(\Delta V)=(900\text{ kg/m}^3)(\pi(9-x^2)\Delta x\text{ m}^3)=900\pi(9-x^2)\Delta x\text{ kg}$$
$$\Delta F=(\Delta m)(a)=(900\pi(9-x^2)\Delta x\text{ kg})(9.8\text{ m/sec}^2)=8820\pi(9-x^2)\Delta x\text{ N}$$
$$\Delta W=(\Delta F)(p)=(8820\pi(9-x^2)\Delta x\text{ N})((x+4)\text{ m})=8820\pi(36+9x-4x^2-x^3)\Delta x\text{ J}$$
$$W=\int_{0}^{3}8820\pi(36+9x-4x^2-x^3)\,dx=8820\pi\left[36x+\frac{9}{2}x^2-\frac{4}{3}x^3-\frac{1}{4}x^4+C\right]_{0}^{3}$$
$$=8820\pi\left\{\left[36(3)+\frac{9}{2}(3)^2-\frac{4}{3}(3)^3-\frac{1}{4}(3)^4+C\right]-\left[36(0)+\frac{9}{2}(0)^2-\frac{4}{3}(0)^3-\frac{1}{4}(0)^4+C\right]\right\}$$
$$=8820\pi\left\{3^2\left[12(1)+\frac{9}{2}-4-\frac{9}{4}\right]-[0]\right\}=8820\pi\left\{3^2\left[8+\frac{18}{4}-\frac{9}{4}\right]\right\}=8820\pi\left\{3^2\left[8+\frac{9}{4}\right]\right\}$$
$$=8820\pi(9)\left(\frac{41}{4}\right)=(9)(41)(2205)\pi=813645\pi\text{ J}$$