

To compute the arc length of a curve we are setting up by cutting the curve into small sections and using hypotenuse of a right triangle as a rate of change and integrating over the interval.

The best formula to memorize is the parametric form, which is given in section 9.2.

If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous on the interval $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β , then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

We must fix the equation given above to be used in section 7.4.

If we have y as a function of x , $y(x)$, then replace all dt by dx in the equation above to get

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2} dx = \int_{\alpha}^{\beta} \sqrt{(1)^2 + \left(\frac{dy}{dx}\right)^2} dx$$

If we have x as a function of y , $x(y)$, then replace all dt by dy in the equation above to get

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dy}\right)^2} dy = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dy}\right)^2 + (1)^2} dy$$

The last type of arc length formula is the polar form, which is introduced in section 9.4.

Given r as a function of θ , the arc length is

$$L = \int_{\alpha}^{\beta} \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Additional Examples:

8) $y^2 = 4(x+4)^3 \quad 0 \leq x \leq 2 \quad y > 0$

$$L = \int_a^b \sqrt{1^2 + \left(\frac{dy}{dx}\right)^2} dx \quad y^2 = 4(x+4)^3 \Rightarrow \begin{aligned} y &= \sqrt{4(x+4)^3} \\ &= 2(x+4)^{\frac{3}{2}} \end{aligned} \quad \frac{dy}{dx} = 3(x+4)^{\frac{1}{2}}(1) = 3\sqrt{x+4}$$

$$\sqrt{1^2 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1^2 + (3\sqrt{x+4})^2} = \sqrt{1+9(x+4)} = \sqrt{9x+37}$$

$$p = 9x+37 \quad dp = 9 dx \quad \frac{1}{9} dp = dx$$

$$\int \sqrt{9x+37} dx = \int \sqrt{p} \left(\frac{1}{9} dp \right) = \frac{1}{9} \left(\frac{2}{3} p^{\frac{3}{2}} \right) + C = \frac{2}{27} (\sqrt{9x+37})^3 + C$$

$$\begin{aligned}
 L &= \int_0^2 \sqrt{9x+37} \, dx = \left[\frac{2}{27} \left(\sqrt{9x+37} \right)^3 + C \right]_0^2 = \left[\frac{2}{27} \left(\sqrt{9(2)+37} \right)^3 + C \right] - \left[\frac{2}{27} \left(\sqrt{9(0)+37} \right)^3 + C \right] \\
 &= \frac{2}{27} \left\{ \left[\left(\sqrt{55} \right)^3 \right] - \left[\left(\sqrt{37} \right)^3 \right] \right\} = \frac{2}{27} \{ 55\sqrt{55} - 37\sqrt{37} \}
 \end{aligned}$$

$$\begin{aligned}
 14) \quad y &= 3 + \frac{1}{2} \cosh 2x \quad 0 \leq x \leq 1 \\
 \frac{dy}{dx} &= \frac{1}{2} (\sinh(2x)(2)) = \sinh(2x) & \cosh^2 \theta - \sinh^2 \theta &= 1 \\
 & & \cosh^2 \theta &= 1 + \sinh^2 \theta \\
 \sqrt{1^2 + \left(\frac{dy}{dx} \right)^2} &= \sqrt{1^2 + (\sinh(2x))^2} = \sqrt{1 + \sinh^2(2x)} = \sqrt{\cosh^2(2x)} = \cosh(2x) \\
 L &= \int_0^1 \cosh(2x) \, dx = \left[\frac{1}{2} \sinh(2x) + C \right]_0^1 = \left[\frac{1}{2} \sinh(2(1)) + C \right] - \left[\frac{1}{2} \sinh(2(0)) + C \right] = \frac{1}{2} \sinh 2
 \end{aligned}$$

$$16) \quad y = \sqrt{x-x^2} + \sin^{-1}(\sqrt{x})$$

The domain for the radical part is $[0,1]$ because for any other values we have complex number. The domain for the $\sin^{-1}(\sqrt{x})$ is also $[0,1]$ because the maximum value of sine is 1. Therefore, our interval is $0 \leq x \leq 1$.

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{1}{2\sqrt{x-x^2}} (1-2x) + \frac{1}{\sqrt{1^2 - (\sqrt{x})^2}} \left(\frac{1}{2\sqrt{x}} \right) = \frac{1-2x}{2\sqrt{(x)(1-x)}} + \frac{1}{2\sqrt{x}\sqrt{1-x}} = \frac{1-2x}{2\sqrt{x}\sqrt{1-x}} + \frac{1}{2\sqrt{x}\sqrt{1-x}} \\
 &= \frac{2-2x}{2\sqrt{x}\sqrt{1-x}} = \frac{2(1-x)}{2\sqrt{x}\sqrt{1-x}} = \frac{\sqrt{1-x}}{\sqrt{x}} = \sqrt{\frac{1-x}{x}} \\
 \sqrt{1^2 + \left(\frac{dy}{dx} \right)^2} &= \sqrt{1^2 + \left(\sqrt{\frac{1-x}{x}} \right)^2} = \sqrt{1 + \frac{1-x}{x}} = \sqrt{\frac{x}{x} + \frac{1-x}{x}} = \sqrt{\frac{1}{x}} = \frac{1}{\sqrt{x}} \\
 L &= \int_0^1 \frac{1}{\sqrt{x}} \, dx = \left[2\sqrt{x} + C \right]_0^1 = \left[2\sqrt{(1)} + C \right] - \left[2\sqrt{(0)} + C \right] = 2
 \end{aligned}$$

$$18) \quad y = 1 + e^{-x} \quad 0 \leq x \leq 2$$

$$\frac{dy}{dx} = e^{-x}(-1) = -e^{-x} = \frac{-1}{e^x}$$

$$\sqrt{1^2 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1^2 + \left(\frac{-1}{e^x}\right)^2} = \sqrt{1 + \frac{1}{e^{2x}}} = \sqrt{\frac{e^{2x}}{e^{2x}} + \frac{1}{e^{2x}}} = \sqrt{\frac{e^{2x} + 1}{e^{2x}}} = \frac{\sqrt{e^{2x} + 1}}{e^x} = \frac{\sqrt{1^2 + (e^x)^2}}{e^x}$$

$$p = e^x \Rightarrow x = \ln p \quad dx = \frac{1}{p} dp$$

$$\int \frac{\sqrt{1^2 + (e^x)^2}}{e^x} dx = \int \frac{\sqrt{1^2 + p^2}}{p} \left(\frac{1}{p} dp \right)$$

$$= \int \frac{\sqrt{1^2 + p^2}}{p^2} dp$$

$$= \int \frac{(\sec \theta)}{(\tan \theta)^2} (\sec^2 \theta d\theta)$$

$$= \int \frac{\sec \theta}{\tan^2 \theta} (1 + \tan^2 \theta) d\theta$$

$$= \int \left(\frac{\sec \theta}{\tan^2 \theta} + \sec \theta \right) d\theta$$

$$= \int \left(\frac{\cos \theta}{\sin^2 \theta} + \sec \theta \right) d\theta$$

$$= \frac{-1}{\sin \theta} + \ln |\sec \theta + \tan \theta| + C = -\csc \theta + \ln |\sec \theta + \tan \theta| + C$$

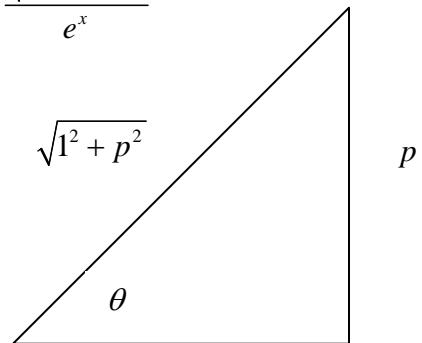
$$= \frac{-\sqrt{1^2 + p^2}}{p} + \ln \left| \sqrt{1^2 + p^2} + p \right| + C = \frac{-\sqrt{1^2 + (e^x)^2}}{e^x} + \ln \left| \sqrt{1^2 + (e^x)^2} + e^x \right| + C$$

$$L = \int_0^2 \frac{\sqrt{1^2 + (e^x)^2}}{e^x} dx = \left[\frac{-\sqrt{1^2 + (e^x)^2}}{e^x} + \ln \left| \sqrt{1^2 + (e^x)^2} + e^x \right| + C \right]_0^2$$

$$= \left[\frac{-\sqrt{1^2 + (e^{(2)})^2}}{e^{(2)}} + \ln \left| \sqrt{1^2 + (e^{(2)})^2} + e^{(2)} \right| + C \right] - \left[\frac{-\sqrt{1^2 + (e^{(0)})^2}}{e^{(0)}} + \ln \left| \sqrt{1^2 + (e^{(0)})^2} + e^{(0)} \right| + C \right]$$

$$= \left[\frac{-\sqrt{1 + e^4}}{e^2} + \ln \left| \sqrt{1 + e^4} + e^2 \right| \right] - \left[\frac{-\sqrt{1 + (1)^2}}{1} + \ln \left| \sqrt{1^2 + (1)^2} + 1 \right| \right]$$

$$= \ln(\sqrt{1 + e^4} + e^2) - \ln(\sqrt{2} + 1) - \frac{\sqrt{1 + e^4}}{e^2} + \sqrt{2}$$



$\frac{p}{1} = \tan \theta$	$\frac{\sqrt{1^2 + p^2}}{1} = \sec \theta$
$p = \tan \theta$	$\frac{1}{\sqrt{1^2 + p^2}} = \sec \theta$
$dp = \sec^2 \theta d\theta$	$\sqrt{1^2 + p^2} = \sec \theta$