

4) $y = x^3$ $y = 0$ $x = 1$ $x = 2$; about y-axis

interval: $1 \leq x \leq 2$

$$r = (x) + 0 = (x)$$

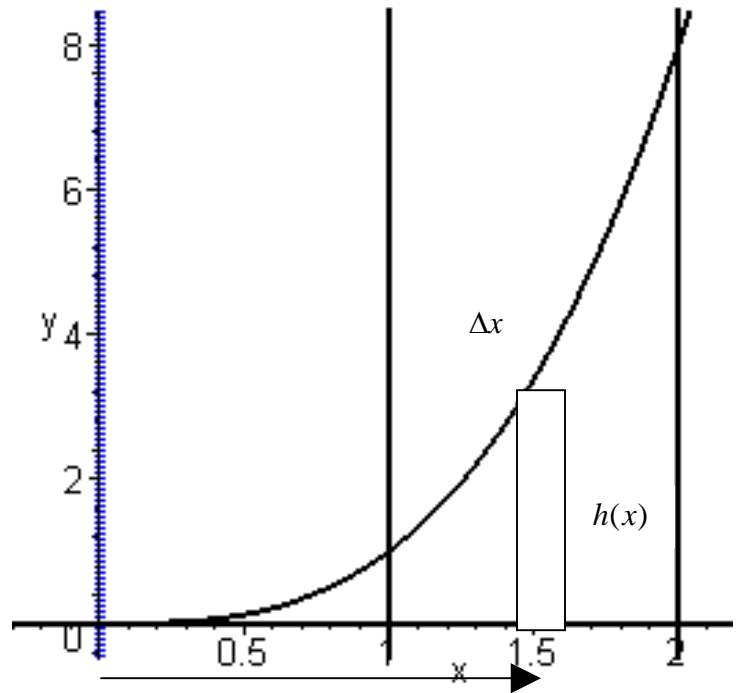
$$l = 2\pi r = 2\pi(x)$$

$$h(x) = (x^3) - (0) = (x^3)$$

$$A = lh(x) = 2\pi(x)(x^3) = 2\pi(x^4)$$

$$\Delta V = A\Delta x = 2\pi(x^4)\Delta x$$

$$\begin{aligned} V &= \int_1^2 2\pi(x^4) dx = 2\pi \left[\frac{1}{5}x^5 + C \right]_1^2 \\ &= 2\pi \left\{ \left[\frac{1}{5}(2)^5 + C \right] - \left[\frac{1}{5}(1)^5 + C \right] \right\} \\ &= 2\pi \left\{ \left[\frac{32}{5} \right] - \left[\frac{1}{5} \right] \right\} = 2\pi \left\{ \frac{31}{5} \right\} = \frac{62\pi}{5} \text{ units}^3 \end{aligned}$$



8) $y = \sqrt{x}$ $y = x^2$; about y-axis

By Shells

interval: $0 \leq x \leq 1$

$$r = (x) + 0 = (x)$$

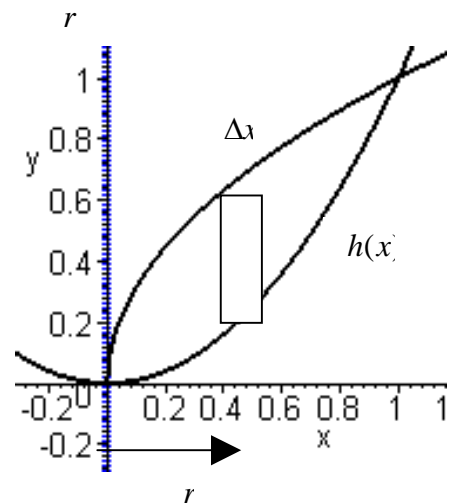
$$l = 2\pi r = 2\pi(x)$$

$$h(x) = (\sqrt{x}) - (x^2) = (\sqrt{x} - x^2)$$

$$A = lh(x) = 2\pi(x)(\sqrt{x} - x^2) = 2\pi(x^{\frac{3}{2}} - x^3)$$

$$\Delta V = A\Delta x = 2\pi(x^{\frac{3}{2}} - x^3)\Delta x$$

$$\begin{aligned} V &= \int_0^1 2\pi(x^{\frac{3}{2}} - x^3) dx = 2\pi \left[\frac{2}{5}x^{\frac{5}{2}} - \frac{1}{4}x^4 + C \right]_0^1 = 2\pi \left[\frac{2}{5}(\sqrt{x})^5 - \frac{1}{4}x^4 + C \right]_0^1 \\ &= 2\pi \left\{ \left[\frac{2}{5}(\sqrt{1})^5 - \frac{1}{4}(1)^4 + C \right] - \left[\frac{2}{5}(\sqrt{0})^5 - \frac{1}{4}(0)^4 + C \right] \right\} = 2\pi \left\{ \frac{2}{5} - \frac{1}{4} \right\} \\ &= 2\pi \left\{ \frac{8}{20} - \frac{5}{20} \right\} = 2\pi \left\{ \frac{3}{20} \right\} = \frac{3\pi}{10} \text{ units}^3 \end{aligned}$$



slicing (washer method)

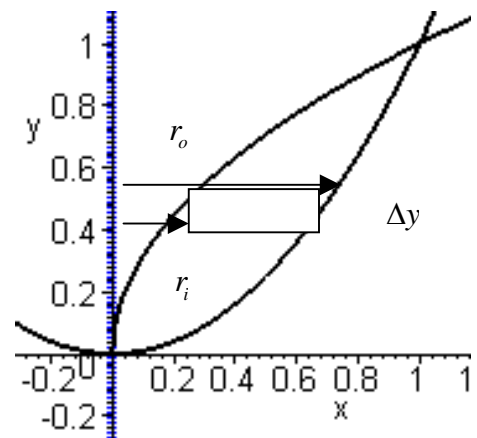
interval: $0 \leq y \leq 1$

$$y = \sqrt{x} \quad y = x^2$$

$$x = y^2 \quad x = \sqrt{y}$$

$$r_o = (\sqrt{y}) + 0 = (\sqrt{y}) \quad r_i = (y^2) + 0 = (y^2)$$

$$A_o = \pi(\sqrt{y})^2 = \pi(y) \quad A_i = \pi(y^2)^2 = \pi(y^4)$$



$$A = A_o - A_i = \pi(y) - \pi(y^4) = \pi(y - y^4) \qquad \Delta V = A\Delta y = \pi(y - y^4)\Delta y$$

$$V = \int_0^1 \pi(y - y^4) dy = \pi \left[\frac{1}{2}y^2 - \frac{1}{5}y^5 + C \right]_0^1 = \pi \left\{ \left[\frac{1}{2}(1)^2 - \frac{1}{5}(1)^5 + C \right] - \left[\frac{1}{2}(0)^2 - \frac{1}{5}(0)^5 + C \right] \right\}$$

$$= \pi \left\{ \frac{1}{2} - \frac{1}{5} \right\} = \pi \left\{ \frac{5}{10} - \frac{2}{10} \right\} = \pi \left\{ \frac{3}{10} \right\} = \frac{3\pi}{10} \text{ units}^3$$

10) $y = \sqrt{x}$ $x = 0$ $y = 2$; about x -axis

$$y = \sqrt{x} \Rightarrow x = y^2$$

interval: $0 \leq y \leq 2$

$$r = (y) + 0 = (y)$$

$$l = 2\pi r = 2\pi(y)$$

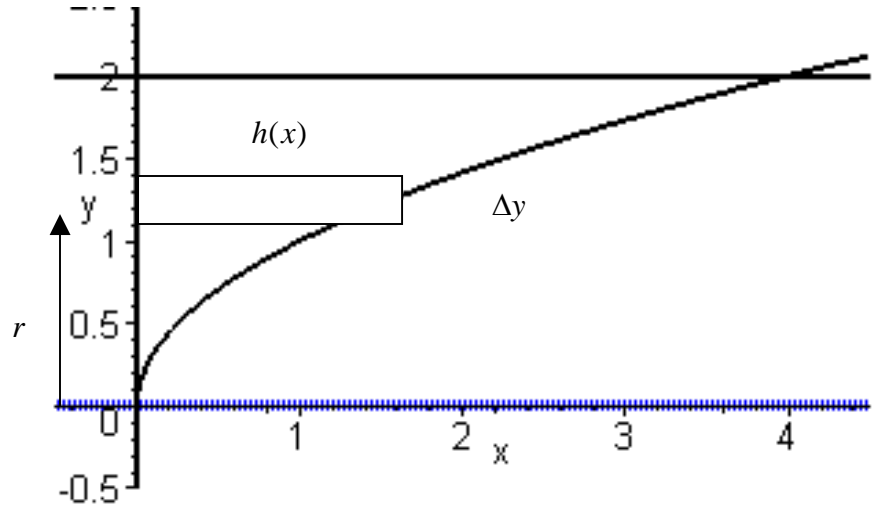
$$h(y) = (y^2) - (0) = (y^2)$$

$$A = lh(y) = 2\pi(y)(y^2) = 2\pi(y^3)$$

$$\Delta V = A\Delta y = 2\pi(y^3)\Delta y$$

$$V = \int_0^2 2\pi(y^3) dy = 2\pi \left[\frac{1}{4}y^4 + C \right]_0^2$$

$$= 2\pi \left\{ \left[\frac{1}{4}(2)^4 + C \right] - \left[\frac{1}{4}(0)^4 + C \right] \right\} = 2\pi\{4\} = 8\pi \text{ units}^3$$



14) $x + y = 3$ $x = 4 - (y - 1)^2$; about x -axis

$$x + y = 3$$

$$x = 3 - y \qquad x = 4 - (y - 1)^2$$

intersect:

$$3 - y = 4 - (y - 1)^2$$

$$3 - y = 4 - (y^2 - 2y + 1) \Rightarrow y(y - 3) = 0$$

$$3 - y = 3 - y^2 + 2y \Rightarrow y = 0 \quad y - 3 = 0$$

$$y^2 - 3y = 0 \qquad y = 3$$

interval: $0 \leq y \leq 3$

$$r = (y) + 0 = (y)$$

$$l = 2\pi r = 2\pi(y)$$

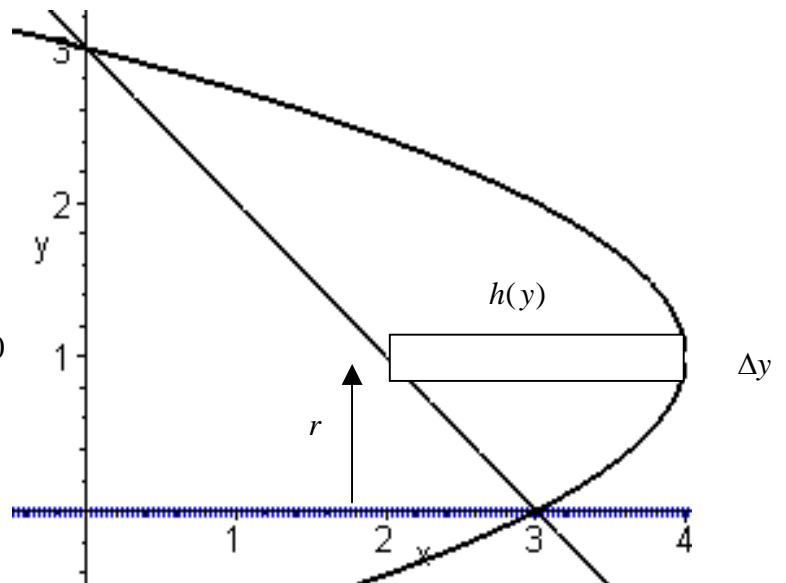
$$h(y) = (4 - (y - 1)^2) - (3 - y)$$

$$= (4 - (y^2 - 2y + 1)) - (3 - y)$$

$$= (3 - y^2 + 2y) - (3 - y) = (3y - y^2)$$

$$A = lh(y) = 2\pi(y)(3y - y^2) = 2\pi(3y^2 - y^3)$$

$$\Delta V = A\Delta y = 2\pi(3y^2 - y^3)\Delta y$$



$$V = \int_0^3 2\pi(3y^2 - y^3) dy = 2\pi \left[y^3 - \frac{1}{4}y^4 + C \right]_0^3 = 2\pi \left\{ \left[(3)^3 - \frac{1}{4}(3)^4 + C \right] - \left[(0)^3 - \frac{1}{4}(0)^4 + C \right] \right\}$$

$$= 2\pi \left\{ (3)^3 \left[1 - \frac{3}{4} \right] - [0] \right\} = 2\pi \left\{ 27 \left[\frac{1}{4} \right] \right\} = \frac{27\pi}{2} \text{ units}^3$$

16) $y = \sqrt{x}$ $y = 0$ $x = 1$; about $x = -1$

interval: $0 \leq x \leq 1$

$$r = (x) + 1 = (x + 1)$$

$$l = 2\pi r = 2\pi(x + 1)$$

$$h(x) = (\sqrt{x}) - (0) = (\sqrt{x})$$

$$A = lh(x) = 2\pi(x + 1)(\sqrt{x}) = 2\pi(x^{\frac{3}{2}} + \sqrt{x})$$

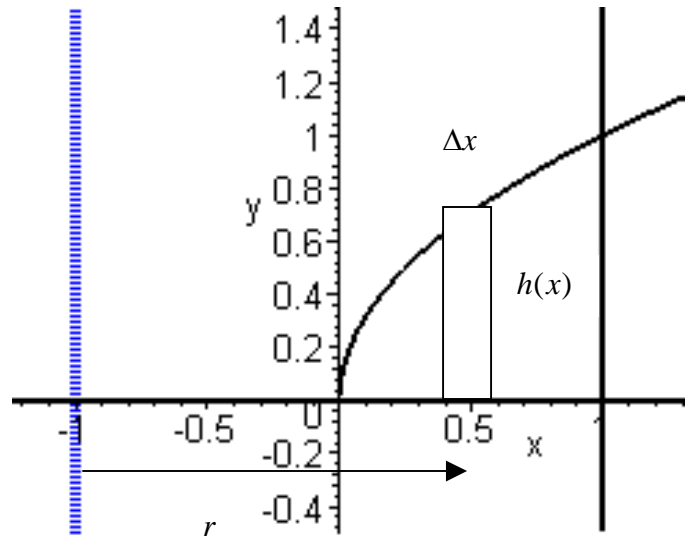
$$\Delta V = A\Delta x = 2\pi(x^{\frac{3}{2}} + \sqrt{x})\Delta x$$

$$V = \int_0^1 2\pi(x^{\frac{3}{2}} + \sqrt{x}) dx = 2\pi \left[\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + C \right]_0^1$$

$$= 2\pi \left[\frac{2}{5}(\sqrt{x})^5 + \frac{2}{3}(\sqrt{x})^3 + C \right]_0^1$$

$$= 2\pi \left\{ \left[\frac{2}{5}(\sqrt{1})^5 + \frac{2}{3}(\sqrt{1})^3 + C \right] - \left[\frac{2}{5}(\sqrt{0})^5 + \frac{2}{3}(\sqrt{0})^3 + C \right] \right\}$$

$$= 2\pi \left\{ \left[\frac{2}{5} + \frac{2}{3} \right] - [0] \right\} = 2\pi \left\{ \frac{6}{15} + \frac{10}{15} \right\} = 2\pi \left\{ \frac{16}{15} \right\} = \frac{32\pi}{15} \text{ units}^3$$



18) $y = x^2$ $y = 2 - x^2$; about $x = 1$

intersect:

$$x^2 = 2 - x^2 \quad 2(x+1)(x-1) = 0$$

$$2x^2 - 2 = 0 \Rightarrow x+1=0 \quad x-1=0$$

$$2(x^2 - 1) = 0 \quad x = -1 \quad x = 1$$

interval: $-1 \leq x \leq 1$

$$r = 1 - (x) = (1 - x)$$

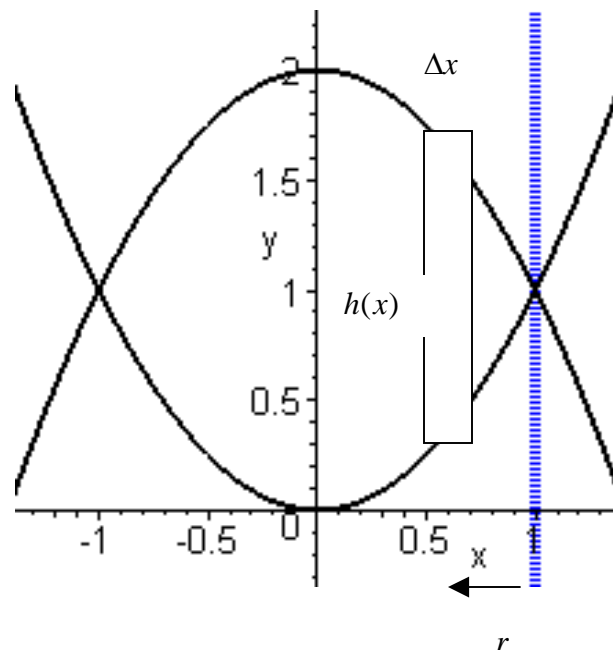
$$l = 2\pi r = 2\pi(1 - x)$$

$$h(x) = (2 - x^2) - (x^2) = (2 - 2x^2)$$

$$A = lh(x) = 2\pi(1 - x)(2 - 2x^2)$$

$$= 2\pi(2 - 2x - 2x^2 + 2x^3)$$

$$\Delta V = A\Delta x = 2\pi(2 - 2x - 2x^2 + 2x^3)\Delta x$$

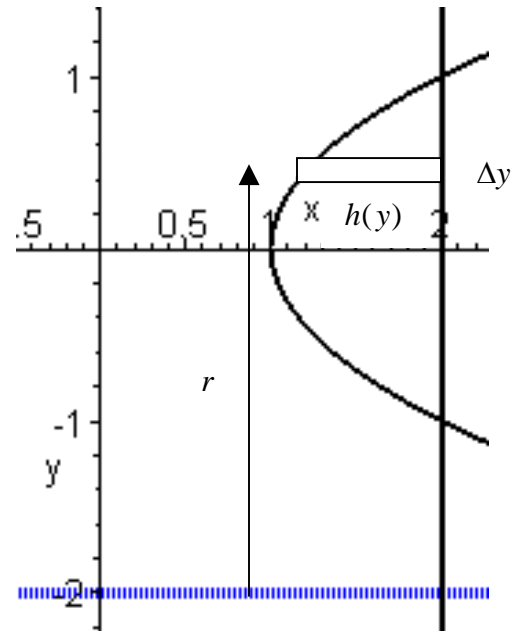


$$\begin{aligned}
 V &= \int_{-1}^1 2\pi(2-2x-2x^2+2x^3) dx = 2\pi \left[2x - x^2 - \frac{2}{3}x^3 + \frac{1}{2}x^4 + C \right]_{-1}^1 \\
 &= 2\pi \left\{ \left[2(1) - (1)^2 - \frac{2}{3}(1)^3 + \frac{1}{2}(1)^4 + C \right] - \left[2(-1) - (-1)^2 - \frac{2}{3}(-1)^3 + \frac{1}{2}(-1)^4 + C \right] \right\} \\
 &= 2\pi \left\{ \left[2 - 1 - \frac{2}{3} + \frac{1}{2} \right] - \left[-2 - 1 + \frac{2}{3} + \frac{1}{2} \right] \right\} = 2\pi \left\{ \left[\frac{3}{2} - \frac{2}{3} \right] - \left[-\frac{5}{2} + \frac{2}{3} \right] \right\} = 2\pi \left\{ \frac{3}{2} - \frac{2}{3} + \frac{5}{2} - \frac{2}{3} \right\} \\
 &= 2\pi \left\{ 4 - \frac{4}{3} \right\} = 2\pi \left\{ \frac{12}{3} - \frac{4}{3} \right\} = 2\pi \left\{ \frac{8}{3} \right\} = \frac{16\pi}{3} \text{ units}^3
 \end{aligned}$$

20) $x = y^2 + 1$ $x = 2$; about $y = -2$

intersect: $2 = y^2 + 1$ $0 = (y+1)(y-1)$
 $0 = y^2 - 1$ $y+1=0$ $y-1=0$
 $y = -1$ $y = 1$

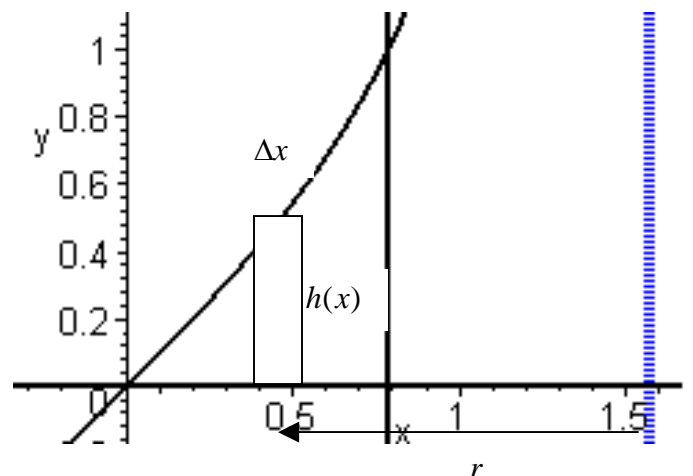
interval: $-1 \leq y \leq 1$
 $r = (y) + 2 = (y + 2)$
 $l = 2\pi r = 2\pi(y + 2)$
 $h(y) = (2) - (y^2 + 1) = (1 - y^2)$
 $A = lh(y) = 2\pi(y + 2)(1 - y^2) = 2\pi(2 + y - 2y^2 - y^3)$
 $\Delta V = A\Delta y = 2\pi(2 + y - 2y^2 - y^3)\Delta y$



$$\begin{aligned}
 V &= \int_{-1}^1 2\pi(2 + y - 2y^2 - y^3) dy = 2\pi \left[2y + \frac{1}{2}y^2 - \frac{2}{3}y^3 - \frac{1}{4}y^4 + C \right]_{-1}^1 \\
 &= 2\pi \left\{ \left[2(1) + \frac{1}{2}(1)^2 - \frac{2}{3}(1)^3 - \frac{1}{4}(1)^4 + C \right] - \left[2(-1) + \frac{1}{2}(-1)^2 - \frac{2}{3}(-1)^3 - \frac{1}{4}(-1)^4 + C \right] \right\} \\
 &= 2\pi \left\{ \left[2 + \frac{1}{2} - \frac{2}{3} - \frac{1}{4} \right] - \left[-2 + \frac{1}{2} + \frac{2}{3} - \frac{1}{4} \right] \right\} = 2\pi \left\{ \left[\frac{9}{4} - \frac{2}{3} \right] - \left[\frac{2}{3} - \frac{7}{4} \right] \right\} \\
 &= 2\pi \left\{ \frac{16}{4} - \frac{4}{3} \right\} = 2\pi \left\{ 4 - \frac{4}{3} \right\} = 2\pi \left\{ \frac{13}{3} - \frac{4}{3} \right\} = 2\pi \left\{ \frac{8}{3} \right\} = \frac{16\pi}{3} \text{ units}^3
 \end{aligned}$$

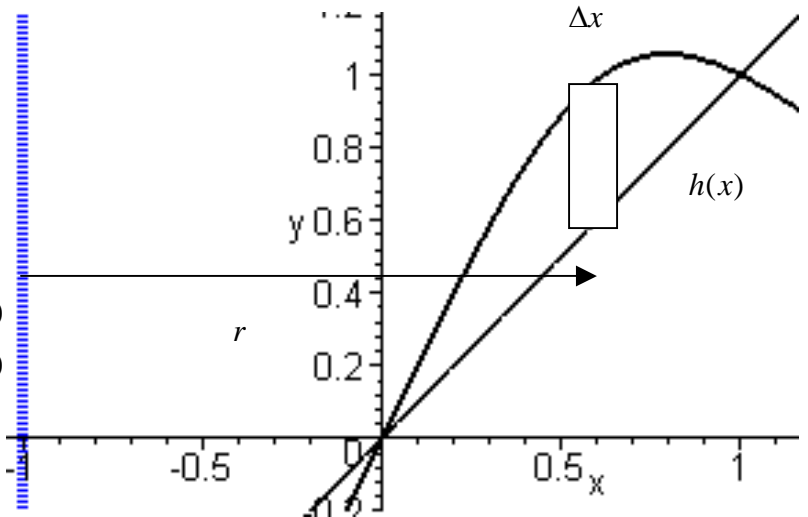
22-a) $y = \tan x$ $y = 0$ $x = \frac{\pi}{4}$; about $x = \frac{\pi}{2}$

interval: $0 \leq x \leq \frac{\pi}{4}$
 $r = \frac{\pi}{2} - (x) = \left(\frac{\pi}{2} - x \right)$ $l = 2\pi r = 2\pi \left(\frac{\pi}{2} - x \right)$
 $h(x) = (\tan x) - (0) = (\tan x)$



$$A = lh(x) = 2\pi \left(\frac{\pi}{2} - x \right) (\tan x) = 2\pi \left(\frac{\pi}{2} \tan x - x \tan x \right) \quad \Delta V = A\Delta x = 2\pi \left(\frac{\pi}{2} \tan x - x \tan x \right) \Delta x$$

$$V = \int_0^{\frac{\pi}{4}} 2\pi \left(\frac{\pi}{2} \tan x - x \tan x \right) dx$$



24-a) $y = x$ $y = \frac{2x}{1+x^3}$; about $x = -1$

intersection:

$$\begin{aligned} x &= \frac{2x}{1+x^3} & x(x^3 - 1) &= 0 \\ x + x^4 &= 2x & x(x-1)(x^2 + x + 1) &= 0 \\ x^4 - x &= 0 & x &= 0 \\ & & x-1 &= 0 \\ & & x &= 1 \end{aligned}$$

interval: $0 \leq x \leq 1$

$$r = (x) + 1 = (x+1) \quad l = 2\pi r = 2\pi(x+1)$$

$$h(x) = \left(\frac{2x}{1+x^3} \right) - (x) = \left(\frac{2x}{1+x^3} - x \right)$$

$$A = lh(x) = 2\pi(x+1) \left(\frac{2x}{1+x^3} - x \right) = 2\pi \left(\frac{2x(x+1)}{1+x^3} - x(x+1) \right) = 2\pi \left(\frac{2x^2}{1+x^3} + \frac{2x}{1+x^3} - x^2 - x \right)$$

$$\Delta V = A\Delta x = 2\pi \left(\frac{2x^2}{1+x^3} + \frac{2x}{1+x^3} - x^2 - x \right) \Delta x \quad V = \int_0^1 2\pi \left(\frac{2x^2}{1+x^3} + \frac{2x}{1+x^3} - x^2 - x \right) dx$$

26-a) $x^2 - y^2 = 7$ $x = 4$; about $y = 5$

$$(4)^2 - y^2 = 7 \quad 0 = (y+3)(y-3)$$

intersection: $16 - y^2 = 7 \Rightarrow y + 3 = 0 \quad y - 3 = 0$

$$0 = y^2 - 9 \quad y = -3 \quad y = 3$$

$$x^2 - y^2 = 7 \quad x^2 = y^2 + 7 \quad x = \pm\sqrt{y^2 + 7} \quad x = \sqrt{y^2 + 7}$$

interval: $-3 \leq y \leq 3$

$$r = 5 - (y) = (5 - y)$$

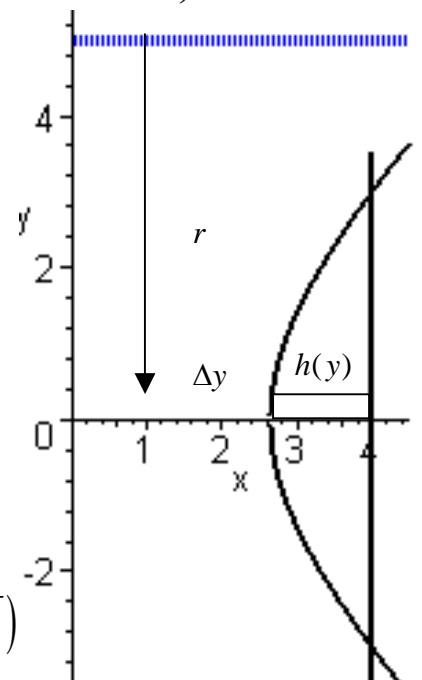
$$l = 2\pi r = 2\pi(5 - y)$$

$$h(y) = 4 - \left(\sqrt{y^2 + 7} \right) = \left(4 - \sqrt{y^2 + 7} \right)$$

$$A = lh(y) = 2\pi(5 - y) \left(4 - \sqrt{y^2 + 7} \right) = 2\pi \left(20 - 4y - 5\sqrt{y^2 + 7} + y\sqrt{y^2 + 7} \right)$$

$$\Delta V = A\Delta y = 2\pi \left(20 - 4y - 5\sqrt{y^2 + 7} + y\sqrt{y^2 + 7} \right) \Delta y$$

$$V = \int_{-3}^3 2\pi \left(20 - 4y - 5\sqrt{y^2 + 7} + y\sqrt{y^2 + 7} \right) dy$$



36) $y^2 - x^2 = 1$ $y = 2$; about y-axis

$$(2)^2 - x^2 = 1$$

$$4 - x^2 = 1$$

$$0 = x^2 - 3$$

intersect:

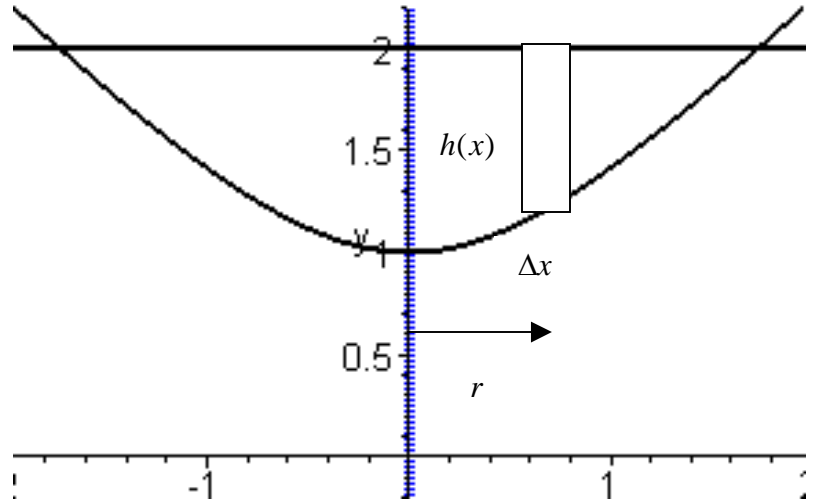
$$0 = (x + \sqrt{3})(x - \sqrt{3})$$

$$x + \sqrt{3} = 0 \quad x - \sqrt{3} = 0$$

$$x = -\sqrt{3} \quad x = \sqrt{3}$$

interval: $0 \leq x \leq \sqrt{3}$

This is from 0 to $\sqrt{3}$ because the graph is symmetric and we don't want to double count the volume.



$$y^2 - x^2 = 1$$

$$y^2 = 1 + x^2 \Rightarrow y = \sqrt{1 + x^2} \quad y = 2$$

$$y = \pm\sqrt{1 + x^2}$$

$$r = (x) + 0 = (x) \quad l = 2\pi r = 2\pi(x)$$

$$h(x) = (2) - (\sqrt{1 + x^2}) = (2 - \sqrt{1 + x^2})$$

$$A = lh(x) = 2\pi(x)(2 - \sqrt{1 + x^2}) = 2\pi(2x - x\sqrt{1 + x^2})$$

$$\Delta V = A\Delta x = 2\pi(2x - x\sqrt{1 + x^2})\Delta x$$

$$V = \int_0^{\sqrt{3}} 2\pi(2x - x\sqrt{1 + x^2}) dx = 2\pi \left[x^2 - \frac{1}{3}(\sqrt{1 + x^2})^3 + C \right]_0^{\sqrt{3}}$$

$$= 2\pi \left\{ \left[(\sqrt{3})^2 - \frac{1}{3}(\sqrt{1 + (\sqrt{3})^2})^3 + C \right] - \left[(0)^2 - \frac{1}{3}(\sqrt{1 + (0)^2})^3 + C \right] \right\}$$

$$= 2\pi \left\{ \left[3 - \frac{1}{3}(2)^3 \right] - \left[0 - \frac{1}{3}(1)^3 \right] \right\} = 2\pi \left\{ 3 - \frac{8}{3} + \frac{1}{3} \right\} = 2\pi \left\{ \frac{9}{3} - \frac{7}{3} \right\} = 2\pi \left\{ \frac{2}{3} \right\} = \frac{4\pi}{3} \text{ units}^3$$

$$p = 1 + x^2 \quad dp = 2x dx \quad \frac{1}{2} dp = x dx$$

because:

$$\int x\sqrt{1 + x^2} dx = \int \sqrt{p} \left(\frac{1}{2} dp \right) = \frac{1}{2} \left(\frac{2}{3} p^{3/2} + C \right) = \frac{1}{3} (\sqrt{1 + x^2})^3 + C$$

40) For this, set the origin in the middle of the tube of torus with center of torus (middle of donut hole) being the rotational axis at $x = -R$

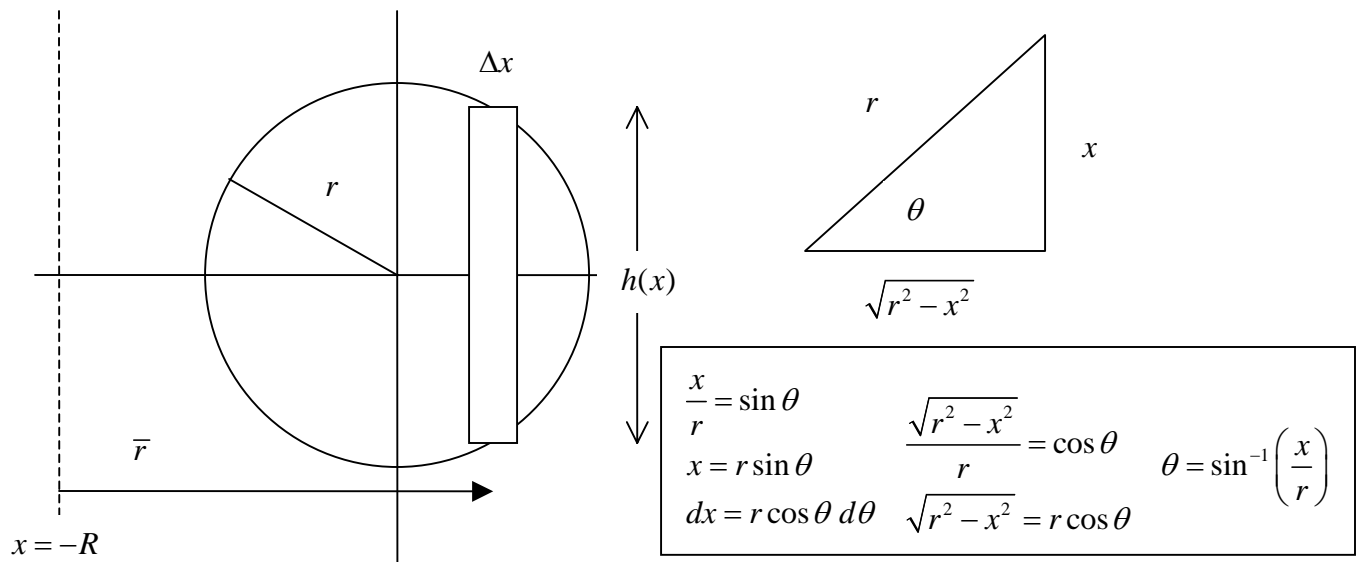
Now the equation of the cross section of torus becomes $x^2 + y^2 = r^2$. Also when we solve this equation

for y, we get

$$x^2 + y^2 = r^2 \Rightarrow y = +\sqrt{r^2 - x^2}$$

$$y = \pm\sqrt{r^2 - x^2} \Rightarrow y = -\sqrt{r^2 - x^2}$$

where positive is upper curve and negative lower one.



interval: $-r \leq x \leq r$ This range is from $-r$ to r because we have a circle of radius r .

$$\bar{r} = (x) + R = (x + R) \quad l = 2\pi\bar{r} = 2\pi(x + R)$$

$$h(x) = \left(+\sqrt{r^2 - x^2} \right) - \left(-\sqrt{r^2 - x^2} \right) = \left(2\sqrt{r^2 - x^2} \right)$$

$$A = lh(x) = 2\pi(x + R) \left(2\sqrt{r^2 - x^2} \right) = 2\pi \left(2x\sqrt{r^2 - x^2} + 2R\sqrt{r^2 - x^2} \right)$$

$$\Delta V = A\Delta x = 2\pi \left(2x\sqrt{r^2 - x^2} + 2R\sqrt{r^2 - x^2} \right) \Delta x$$

$$\begin{aligned} \int \sqrt{r^2 - x^2} \, dx &= \int (r \cos \theta)(r \cos \theta \, d\theta) = \int r^2 \cos^2 \theta \, d\theta = \int r^2 \left(\frac{1}{2}(1 + \cos 2\theta) \right) d\theta = \frac{r^2}{2} \left(\theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{r^2}{2} (\theta + \sin \theta \cos \theta) + C = \frac{r^2}{2} \left(\sin^{-1} \left(\frac{x}{r} \right) + \left(\frac{x}{r} \right) \left(\frac{\sqrt{r^2 - x^2}}{r} \right) \right) + C = \frac{r^2}{2} \sin^{-1} \left(\frac{x}{r} \right) + \frac{1}{2} x \sqrt{r^2 - x^2} + C \end{aligned}$$

$$\begin{aligned} V &= \int_{-r}^r 2\pi \left(2x\sqrt{r^2 - x^2} + 2R\sqrt{r^2 - x^2} \right) dx = 2\pi \left\{ \int_{-r}^r 2x\sqrt{r^2 - x^2} \, dx + \int_{-r}^r 2R\sqrt{r^2 - x^2} \, dx \right\} \\ &= 2\pi \left\{ \left[-\left(\sqrt{r^2 - x^2} \right)^3 + C \right]_{-r}^r + 2R \left[\frac{r^2}{2} \sin^{-1} \left(\frac{x}{r} \right) + \frac{1}{2} x \sqrt{r^2 - x^2} + C \right]_{-r}^r \right\} \\ &= 2\pi \left\{ \left\{ \left[-\left(\sqrt{r^2 - (r)^2} \right)^3 + C \right] - \left[-\left(\sqrt{r^2 - (-r)^2} \right)^3 + C \right] \right\} \right. \\ &\quad \left. + 2R \left\{ \left[\frac{r^2}{2} \sin^{-1} \left(\frac{(r)}{r} \right) + \frac{1}{2} (r) \sqrt{r^2 - (r)^2} + C \right] - \left[\frac{r^2}{2} \sin^{-1} \left(\frac{(-r)}{r} \right) + \frac{1}{2} (-r) \sqrt{r^2 - (-r)^2} + C \right] \right\} \right\} \\ &= 2\pi \left\{ \{ [0] - [0] \} + 2R \left\{ \left[\frac{r^2}{2} \sin^{-1} (1) + 0 \right] - \left[\frac{r^2}{2} \sin^{-1} (-1) + 0 \right] \right\} \right\} \\ &= 2\pi \left\{ \{ 0 \} + 2R \left\{ \left[\frac{r^2}{2} \left(\frac{\pi}{2} \right) \right] - \left[\frac{r^2}{2} \left(\frac{-\pi}{2} \right) \right] \right\} \right\} \\ &= 2\pi \left\{ 2R \left\{ \left[\frac{\pi r^2}{4} \right] - \left[\frac{-\pi r^2}{4} \right] \right\} \right\} = 2\pi \left\{ 2R \left\{ \frac{\pi r^2}{2} \right\} \right\} = 2\pi^2 r^2 R \text{ units}^3 \end{aligned}$$