Definition of Improper integrals

Type 1:

(a) $\int_a^\infty f(x)\,dx = \lim_{U\to\infty}\int_a^U f(x)\,dx$

(b) $\int_{-\infty}^b f(x)\,dx = \lim_{L\to-\infty}\int_L^b f(x)\,dx$

(c) $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$ for any real number c

Type 2:

(a) If f is continuous on $[a,b)$ and is discontinuous at b, then $\int_a^b f(x)\,dx = \lim_{U\to b^-}\int_a^U f(x)\,dx$

(b) If f is continuous on $(a,b]$ and is discontinuous at a, then $\int_a^b f(x)\,dx = \lim_{L\to a^+}\int_L^b f(x)\,dx$

(c) If f has a discontinuity at c, where $a < c < b$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

Comparison Theorem: Suppose that f and g are continuous functions with $f(x) \ge g(x) \ge 0$ for $x \ge a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Reference for Comparison Theorem:

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \text{convergent if} & p > 1 \\ \text{divergent if} & p \le 1 \end{cases} \qquad \int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \text{divergent if} & p \ge 1 \\ \text{convergent if} & p < 1 \end{cases}$$

Additional examples:

10) $\int_{-\infty}^0 2^r\,dr$

$$\int 2^r\,dr = \frac{1}{\ln 2}2^r + C$$

$$\int_{-\infty}^0 2^r\,dr = \lim_{L\to-\infty}\int_L^0 2^r\,dr = \lim_{L\to-\infty}\left[\frac{1}{\ln 2}2^r + C\right]_L^0 = \lim_{L\to-\infty}\left\{\left[\frac{1}{\ln 2}2^{(0)} + C\right] - \left[\frac{1}{\ln 2}2^L + C\right]\right\}$$

$$= \left[\frac{1}{\ln 2}(1)\right] - \left[\frac{1}{\ln 2}(0)\right] = \frac{1}{\ln 2}$$

Convergent or Converges

14) $\int_{-\infty}^{\infty} x^2 e^{-x^3}\, dx$

$$p = -x^3 \quad dp = -3x^2\, dx \quad \frac{-1}{3}\,dp = x^2\, dx$$

$$\int x^2 e^{-x^3}\, dx = \int e^p \left(\frac{-1}{3}\,dp\right) = \frac{-1}{3} e^p + C = \frac{-1}{3} e^{-x^3} + C = \frac{-1}{3e^{x^3}} + C$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^3}\, dx = \int_{-\infty}^{0} x^2 e^{-x^3}\, dx + \int_{0}^{\infty} x^2 e^{-x^3}\, dx$$

$$\int_{0}^{\infty} x^2 e^{-x^3}\, dx = \lim_{U\to\infty}\int_{0}^{U} x^2 e^{-x^3}\, dx = \lim_{U\to\infty}\left[\frac{-1}{3e^{x^3}} + C\right]_0^U = \lim_{U\to\infty}\left\{\left[\frac{-1}{3e^{U^3}} + C\right] - \left[\frac{-1}{3e^{(0)^3}} + C\right]\right\} = \frac{1}{3}$$

$$\int_{-\infty}^{0} x^2 e^{-x^3}\, dx = \lim_{L\to-\infty}\int_{L}^{0} x^2 e^{-x^3}\, dx = \lim_{L\to-\infty}\left[\frac{-1}{3e^{x^3}} + C\right]_L^0 = \lim_{L\to-\infty}\left\{\left[\frac{-1}{3e^{(0)^3}} + C\right] - \left[\frac{-1}{3e^{L^3}} + C\right]\right\} = \infty$$

Divergent or Diverges

16) $\int_{-\infty}^{\infty} \cos \pi t\, dt$

$$\int \cos \pi t\, dt = \frac{1}{\pi}\sin \pi t + C$$

$$\int_{-\infty}^{\infty} \cos \pi t\, dt = \int_{-\infty}^{0} \cos \pi t\, dt + \int_{0}^{\infty} \cos \pi t\, dt$$

$$\int_{0}^{\infty} \cos \pi t\, dt = \lim_{U\to\infty}\int_{0}^{U} \cos \pi t\, dt = \lim_{U\to\infty}\left[\frac{1}{\pi}\sin \pi t + C\right]_0^U = \lim_{U\to\infty}\left\{\left[\frac{1}{\pi}\sin \pi U + C\right] - \left[\frac{1}{\pi}\sin \pi(0) + C\right]\right\} = \text{DNE}$$

Limit does not exist for $\sin(\pi U)$ as $U \to \infty$. Therefore, the limit is divergent, thus making $\int_{-\infty}^{\infty} \cos \pi t\, dt$ divergent.

20) $\int_{1}^{\infty} \frac{\ln x}{x^3}\, dx$

$$\int \frac{\ln x}{x^3}\, dx = (\ln x)\left(\frac{-1}{2x^2}\right) - \int \left(\frac{-1}{2x^2}\right)\left(\frac{1}{x}\, dx\right) \longleftarrow \quad u_1 = \ln x \quad dv_1 = \frac{1}{x^3}\, dx$$

$$= \frac{-\ln x}{2x^2} + \int \frac{1}{2x^3}\, dx = \frac{-\ln x}{2x^2} - \frac{1}{4x^2} + C \qquad du_1 = \frac{1}{x}\, dx \quad v_1 = \frac{-1}{2x^2}$$

$$\int_{1}^{\infty} \frac{\ln x}{x^3}\, dx = \lim_{U\to\infty}\int_{1}^{U} \frac{\ln x}{x^3}\, dx = \lim_{U\to\infty}\left[\frac{-\ln x}{2x^2} - \frac{1}{4x^2} + C\right]_1^U$$

Converges

$$= \lim_{U\to\infty}\left\{\left[\frac{-\ln U}{2U^2} - \frac{1}{4U^2} + C\right] - \left[\frac{-\ln(1)}{2(1)^2} - \frac{1}{4(1)^2} + C\right]\right\} = \frac{1}{4}$$

because: $\lim_{U\to\infty} \frac{\overset{-\infty}{-\ln U}}{\underset{\infty}{2U^2}} \overset{L}{=} \lim_{U\to\infty} \frac{\frac{-1}{U}}{4U} = \lim_{U\to\infty} \frac{-1}{4U^2} = 0$

22) $\displaystyle\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx$

$$p = e^x \qquad dp = e^x\ dx$$

$$\int \frac{e^x}{e^{2x}+3}\,dx = \int \frac{e^x}{(e^x)^2+3}\,dx = \int \frac{p}{p^2+(\sqrt{3})^2}\,dx = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{p}{\sqrt{3}}\right)+C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{e^x}{\sqrt{3}}\right)+C$$

$$\int_0^\infty \frac{e^x}{e^{2x}+3}\,dx = \lim_{U\to\infty}\int_0^U \frac{e^x}{e^{2x}+3}\,dx = \lim_{U\to\infty}\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{e^x}{\sqrt{3}}\right)+C\right]_0^U$$

$$= \lim_{U\to\infty}\left\{\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{e^U}{\sqrt{3}}\right)+C\right]-\left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{e^{(0)}}{\sqrt{3}}\right)+C\right]\right\} = \left[\frac{1}{\sqrt{3}}\left(\frac{\pi}{2}\right)\right]-\left[\frac{1}{\sqrt{3}}\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{3\pi}{6\sqrt{3}}-\frac{\pi}{6\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$$

Converges

26) $\displaystyle\int_6^8 \frac{4}{(x-6)^3}\,dx$

$$\int \frac{4}{(x-6)^3}\,dx = 4\left(\frac{-1}{2(x-6)^2}\right)+C = \frac{-2}{(x-6)^2}+C$$

$$\int_6^8 \frac{4}{(x-6)^3}\,dx = \lim_{L\to6^+}\int_L^8 \frac{4}{(x-6)^3}\,dx = \lim_{L\to6^+}\left[\frac{-2}{(x-6)^2}+C\right]_L^8 = \lim_{L\to6^+}\left\{\left[\frac{-2}{((8)-6)^2}+C\right]-\left[\frac{-2}{(L-6)^2}+C\right]\right\} = \infty$$

Diverges

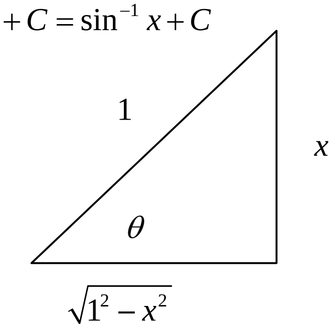
30) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{1}{\sqrt{1^2-x^2}}\,dx = \int \frac{1}{(\cos\theta)}(\cos\theta\ d\theta) = \int 1\,d\theta = \theta + C = \sin^{-1}\left(\frac{x}{1}\right)+C = \sin^{-1}x + C$$

$$\frac{x}{1} = \sin\theta \qquad \frac{\sqrt{1^2-x^2}}{1} = \cos\theta \qquad \theta = \sin^{-1}\left(\frac{x}{1}\right)$$

$$x = \sin\theta \qquad \sqrt{1^2-x^2} = \cos\theta$$

$$dx = \cos\theta\ d\theta$$



$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{U\to1^-}\int_0^U \frac{dx}{\sqrt{1-x^2}} = \lim_{U\to1^-}\left[\sin^{-1}x + C\right]_0^U$$

$$= \lim_{U\to1^-}\left\{\left[\sin^{-1}U + C\right]-\left[\sin^{-1}(0)+C\right]\right\} = \frac{\pi}{2}$$

Converges

32) $\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = (\ln x)(2\sqrt{x}) - \int (2\sqrt{x})\left(\frac{1}{x}\,dx\right)$$

$u_1 = \ln x \qquad dv_1 = \frac{1}{\sqrt{x}}\,dx$

$du_1 = \frac{1}{x}\,dx \qquad v_1 = 2\sqrt{x}$

$$= 2\sqrt{x}\ln x - \int \frac{2}{\sqrt{x}}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C$$

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{L\to 0^+}\int_L^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{L\to 0^+}\Big[2\sqrt{x}\ln x - 4\sqrt{x} + C\Big]_L^1$$

$$= \lim_{L\to 0^+}\left\{\Big[2\sqrt{(1)}\ln(1) - 4\sqrt{(1)} + C\Big] - \Big[2\sqrt{L}\ln L - 4\sqrt{L} + C + C\Big]\right\} = -4$$

Converges

because $\lim_{L\to 0^+} \overset{0}{2\sqrt{L}}\,\overset{-\infty}{\ln L} = \lim_{L\to 0^+} \frac{\overset{-\infty}{2\ln L}}{\underset{\infty}{\frac{1}{\sqrt{L}}}} \overset{L}{=} \lim_{L\to 0^+} \frac{\frac{2}{L}}{\frac{-1}{2}L^{\frac{-3}{2}}} = \lim_{L\to 0^+} \frac{\frac{2}{L}}{\frac{-1}{}} = \lim_{L\to 0^+}\left(\frac{2}{L}\right)\left(\frac{2(\sqrt{L})^3}{-1}\right) = \lim_{L\to 0^+} -4\sqrt{L} = 0$

46) $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$

For $0 \le x \le 1,\ 0 < \frac{\sin^2 x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$

$$\int_0^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{L\to 0^+}\int_L^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{L\to 0^+}\Big[2\sqrt{x} + C\Big]_L^\pi = \lim_{L\to 0^+}\left\{\Big[2\sqrt{(\pi)} + C\Big] - \Big[2\sqrt{L} + C\Big]\right\}$$

$$= \Big[2\sqrt{\pi}\Big] - \Big[2\sqrt{(0^+)}\Big] = 2\sqrt{\pi}$$

Converges and thus $\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$ converges