Integration of Rational Function by Partial Fractions

If $f(x)=\dfrac{P(x)}{Q(x)}$ such that $\deg(P(x))\geq\deg(Q(x))$, then use the long division to obtain
$f(x)=\dfrac{P(x)}{Q(x)}=S(x)+\dfrac{R(x)}{Q(x)}$ where $S(x)$ and $R(x)$ are polynomials.

Case 1: The denominator $Q(x)$ is a product of distinct linear factors.
This means that we can write $Q(x)=(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$ where no factor is repeated. In this case the partial fraction theorem states that there exist constants $A_1, A_2, \ldots, A_k$ such that
$\dfrac{R(x)}{Q(x)}=\dfrac{A_1}{a_1x+b_1}+\dfrac{A_2}{a_2x+b_2}+\cdots+\dfrac{A_k}{a_kx+b_k}$.

Case 2: $Q(x)$ is a product of linear factors, some which are repeated.
Suppose the first linear factor (a_1x+b_1) is repeated r times; that is, $(a_1x+b_1)^r$ occurs in the factorization of $Q(x)$. Then instead of the single term $\dfrac{A_1}{(a_1x+b_1)}$ in previous case 1, we would use
$\dfrac{A_1}{(a_1x+b_1)}+\dfrac{A_2}{(a_2x+b_2)^2}+\cdots+\dfrac{A_r}{(a_rx+b_r)^r}$.

Case 3: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.
If $Q(x)$ has the factor ax^2+bx+c, where $b^2-4ac<0$, then, in addition to the partial fractions in equations from case 1 and 2, the expression for $\dfrac{R(x)}{Q(x)}$ will have a term of the form $\dfrac{Ax+B}{ax^2+bx+c}$ where A and B are constants to be determined.
The term $\dfrac{Ax+B}{ax^2+bx+c}$ can be integrated by completing the square and using the formula
$\displaystyle\int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$.

Case 4: $Q(x)$ contains a repeated irreducible quadratic factor.
If $Q(x)$ has the factor $(ax^2+bx+c)^r$, where $b^2-4ac<0$, then instead of the single partial fraction in case 3, the sum $\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\dfrac{A_rx+B_r}{\left(ax^2+bx+c\right)^r}$ occurs in the partial fraction decomposition of $\dfrac{R(x)}{Q(x)}$. Each of the terms above can be integrated by first completing the square.

The work in this section is based on Algebra. Depending on question, one may need to do many manipulations and simplifications to set up the integration correctly. Make sure to brush up on algebraic skills.

Additional examples:

6-a) $\dfrac{t^6+1}{t^6+t^3}$

$$\frac{t^6+1}{t^6+t^3}=1+\frac{(-t^3+1)}{t^6+t^3}=1+\frac{(-t^3+1)}{t^3(t^3+1)}=1+\frac{(-t^3+1)}{t^3(t+1)(t^2+t+1)}$$

because $(A^3+B^3)=(A+B)(A^2-AB+B^2)$

$$\begin{array}{r} 1 \\ t^6+0t^5+0t^4+1t^3+0t^2+0t+0\overline{\big)\,t^6+0t^5+0t^4+0t^3+0t^2+0t+1} \\ -\underline{(t^6+0t^5+0t^4+t^3+0t^2+0t+0)} \\ -t^3+1 \end{array}$$

$$\frac{(-t^3+1)}{(t)^3(t+1)^1(t^2+t+1)^1}=\frac{A}{(t)^1}+\frac{B}{(t)^2}+\frac{C}{(t)^3}+\frac{D}{(t+1)^1}+\frac{(Ex+F)}{(t^2+t+1)^1}$$

$$\frac{t^6+1}{t^6+t^3}=1+\frac{(-t^3+1)}{t^6+t^3}=1+\frac{(-t^3+1)}{t^3(t^3+1)}=1+\frac{(-t^3+1)}{t^3(t+1)(t^2+t+1)}=1+\frac{A}{(t)^1}+\frac{B}{(t)^2}+\frac{C}{(t)^3}+\frac{D}{(t+1)^1}+\frac{(Ex+F)}{(t^2+t+1)^1}$$

6-b) $\dfrac{x^5+1}{(x^2-x)(x^4+2x^2+1)}$

$$\frac{x^5+1}{(x^2-x)(x^4+2x^2+1)}=\frac{x^5+1}{x(x-1)(x^2+1)(x^2+1)}$$

$$=\frac{x^5+1}{(x)^1(x-1)^1(x^2+1)^2}=\frac{A}{(x)^1}+\frac{B}{(x-1)^1}+\frac{(Cx+D)}{(x^2+1)^1}+\frac{(Ex+F)}{(x^2+1)^2}$$

10) $\displaystyle\int\frac{y}{(y+4)(2y-1)}\,dy$

$$\frac{y}{(y+4)^1(2y-1)^1}=\frac{A}{(y+4)^1}+\frac{B}{(2y-1)^1}$$

$$y=A(2y-1)+B(y+4)$$

const term
$0=-A+4B$
$A=4B$

$y-term$
$1=2A+B$

$1=2(4B)+B$
$1=9B$
$\frac{1}{9}=B$

$A=4\left(\frac{1}{9}\right)=\frac{4}{9}$

$$\int\frac{y}{(y+4)(2y-1)}\,dy=\int\left(\frac{(\frac{4}{9})}{y+4}+\frac{(\frac{1}{9})}{2y-1}\right)dy=\int\left(\frac{\frac{4}{9}}{y+4}+\frac{\frac{1}{9}}{2(y-\frac{1}{2})}\right)dy=\int\left(\frac{\frac{4}{9}}{y+4}+\frac{\frac{1}{18}}{y-\frac{1}{2}}\right)dy$$

$$=\frac{4}{9}\ln|y+4|+\frac{1}{18}\ln\left|y-\frac{1}{2}\right|+C$$

12) $\int_0^1 \frac{x-4}{x^2-5x+6}\,dx$

$$\frac{x-4}{x^2-5x+6} = \frac{x-4}{(x-3)^1(x-2)^1} = \frac{A}{(x-3)^1} + \frac{B}{(x-2)^1}$$

$$x-4 = A(x-2)+B(x-3)$$

const term	x − term		
$-4=-2A-3B$	$1=A+B$	$-4=-2A-3(1-A)$	$B=1-(-1)$
	$B=(1-A)$	$-4=A-3$	$B=2$
		$-1=A$	

$$\int \frac{x-4}{x^2-5x+6}\,dx = \int\left(\frac{(-1)}{(x-3)}+\frac{(2)}{(x-2)}\right)dx = -\ln|x-3|+2\ln|x-2|+C = 2\ln|x-2|-\ln|x-3|+C$$

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,dx = \Big[2\ln|x-2|-\ln|x-3|+C\Big]_0^1$$

$$= \Big[2\ln|(1)-2|-\ln|(1)-3|+C\Big]-\Big[2\ln|(0)-2|-\ln|(0)-3|+C\Big]$$

$$= \Big[2\ln|-1|-\ln|-2|\Big]-\Big[2\ln|-2|-\ln|-3|\Big] = \Big[2(0)-\ln 2\Big]-\Big[2\ln 2-\ln 3\Big]$$

$$= \ln 3 - 3\ln 2 = \ln 3 - \ln(2^3) = \ln 3 - \ln 8 = \ln\left(\frac{3}{8}\right)$$

16) $\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx$

$$\begin{array}{r|l} & \quad x+1 \\ x^2-x-6 & x^3+0x^2-4x-10 \\ & -(x^3-x^2-6x) \\ & \quad 1x^2+2x-10 \\ & -(x^2-x-6) \\ & \quad 3x-4 \end{array}$$

$$\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{(3x-4)}{x^2-x-6}$$

$$\frac{(3x-4)}{(x+2)^1(x-3)^1} = \frac{A}{(x+2)^1}+\frac{B}{(x-3)^1}$$

$$3x-4 = A(x-3)+B(x+2)$$

const. term	x term
$-4=-3A+2B$	$3=A+B$
$-4=-3A+2(3-A)$	$3-A=B$
$-10=-5A$	$3-(2)=B$
$2=A$	$1=B$

$$\int \frac{x^3-4x-10}{x^2-x-6}\,dx = \int\left(x+1+\frac{(3x-4)}{x^2-x-6}\right)dx = \int\left(x+1+\frac{(2)}{(x+2)}+\frac{(1)}{(x-3)}\right)dx$$

$$= \frac{1}{2}x^2+x+2\ln|x+2|+\ln|x-3|+C$$

$$\int_0^1 \frac{x^3-4x-10}{x^2-x-6}\,dx = \left[\frac{1}{2}x^2+x+2\ln|x+2|+\ln|x-3|+C\right]_0^1$$

$$= \left[\frac{1}{2}(1)^2+(1)+2\ln|(1)+2|+\ln|(1)-3|+C\right]-\left[\frac{1}{2}(0)^2+(0)+2\ln|(0)+2|+\ln|(0)-3|+C\right]$$

$$= \left[\frac{3}{2}+2\ln|3|+\ln|-2|\right]-\Big[0+2\ln|2|+\ln|-3|\Big] = \frac{3}{2}+\ln(3)-\ln(2) = \frac{3}{2}\ln\left(\frac{3}{2}\right)$$

18) $\int \frac{x^2+2x-1}{x^3-x}\,dx$

$$\frac{x^2+2x-1}{x^3-x}=\frac{x^2+2x-1}{(x)^1(x+1)^1(x-1)^1}=\frac{A}{(x)^1}+\frac{B}{(x+1)^1}+\frac{C}{(x-1)^1}$$

$$x^2+2x-1=A(x+1)(x-1)+B(x(x-1))+C(x(x+1))$$

$$x^2+2x-1=A(x^2-1)+B(x^2-x)+C(x^2+x)$$

const term
$-1=-A$
$A=1$

$x-term$
$-2=-B+C$

x^2-term
$1=A+B+C$
$1=(1)+B+C$
$0=B+C$
$C=-B$

$-2=(C)+C$
$-2=2C$
$-1=C$

$-B=-1$
$B=1$

$$\int \frac{x^2+2x-1}{x^3-x}\,dx=\int\left(\frac{(1)}{(x)}+\frac{(1)}{(x+1)}+\frac{(-1)}{(x-1)}\right)dx$$

$$=\ln|x|+\ln|x+1|-\ln|x-1|+C$$

26) $\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx$

$$\frac{x^3-2x^2+x+1}{x^4+5x^2+4}=\frac{x^3-2x^2+x+1}{(x^2+1)^1(x^2+4)^1}=\frac{(Ax+B)}{(x^2+1)^1}+\frac{(Cx+D)}{(x^2+4)^1}$$

$$x^3-2x^2+x+1=(Ax+B)(x^2+4)+(Cx+D)(x^2+1)$$

$$x^3-2x^2+x+1=A(x^3+4x)+B(x^2+4)+C(x^3+x)+D(x^2+1)$$

const term
$1=4B+D$

$x-term$
$1=4A+C$

x^2-term
$-2=B+D$
$D=(-B-2)$

x^3-term
$1=A+C$
$C=1-A$

$1=4B+(-B-2)$
$3=3B$
$B=1$

$1=4A+(1-A)$
$0=3A$
$A=0$

$D=-(1)-2$
$D=-3$

$C=1-(0)$
$C=1$

$$\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx=\int\left(\frac{(0)x+(1)}{(x^2+1)^1}+\frac{(1)x+(-3)}{(x^2+4)^1}\right)dx$$

$$=\int\left(\frac{(1)}{x^2+1^2}+\frac{(1)x}{(x^2+4)}+\frac{(-3)}{x^2+2^2}\right)dx$$

$$=\frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right)+\frac{1}{2}\ln|x^2+4|+(-3)\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right)+C$$

$$=\tan^{-1}x+\frac{1}{2}\ln|x^2+4|-\frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$$

24) $\displaystyle\int \frac{x^2-2x-1}{(x-2)^2(x^2+1)}\,dx$

$$\frac{x^2-2x-1}{(x-2)^2(x^2+1)^1}=\frac{A}{(x-2)^1}+\frac{B}{(x-2)^2}+\frac{(Cx+D)}{(x^2+1)^1}$$

$$x^2-2x-1=A(x-1)(x^2+1)+B(x^2+1)+(Cx+D)(x-2)^2$$

$$x^2-2x-1=A(x^3-2x^2+x-2)+B(x^2+1)+(Cx+D)(x^2-4x+4)$$

$$x^2-2x-1=A(x^3-2x^2+x-2)+B(x^2+1)+C(x^3-4x^2+4x)+D(x^2-4x+4)$$

const term

$$-1=-2A+B+4D$$
$$B=(2A-4D-1)$$

$x-term$

$$-2=A+4C-4D$$
$$-2=A+4(-A)-4D$$
$$-2=-3A-4D$$
$$2=3A+4D$$

x^2-term

$$1=-2A+B-4C+D$$
$$1=-2A+B-4(-A)+D$$
$$1=2A+B+D$$

x^3-term

$$0=A+C$$
$$C=-A$$

$$1=2A+(2A-4D-1)+D$$
$$2=4A-3D$$

$$2=3A+4D\xrightarrow{3}\ 6=9A+12D$$
$$2=4A-3D\xrightarrow{4}\ 8=16A-12D$$

$$\Rightarrow\quad 14=25A,\quad A=\frac{14}{25}\quad\Rightarrow\quad C=\frac{-14}{25}$$

$$2=3\left(\frac{14}{25}\right)+4D$$
$$2=\frac{42}{25}+4D$$
$$4D=2-\frac{42}{25}$$
$$4D=\frac{8}{25}$$
$$D=\frac{2}{25}$$

$$B=2\left(\frac{14}{25}\right)-4\left(\frac{2}{25}\right)-1$$
$$B=\frac{28}{25}-\frac{8}{25}-\frac{25}{25}$$
$$B=\frac{20}{25}-\frac{25}{25}$$
$$B=\frac{-5}{25}=\frac{-1}{5}$$

$$\int \frac{x^2-2x-1}{(x-2)^2(x^2+1)}\,dx=\int\left(\frac{(\frac{14}{25})}{(x-2)}+\frac{(\frac{-1}{5})}{(x-2)^2}+\frac{(\frac{-14}{25})x+(\frac{2}{25})}{(x^2+1)}\right)dx$$

$$=\int\left(\frac{(\frac{14}{25})}{(x-2)}+\frac{(\frac{-1}{5})}{(x-2)^2}+\frac{(\frac{-14}{25})x}{(x^2+1)}+\frac{(\frac{2}{25})}{(x^2+1^2)}\right)dx$$

$$=\frac{14}{25}\ln|x-2|+\left(\frac{-1}{5}\right)\left(\frac{-1}{(x-2)^1}\right)+\left(\frac{-14}{25}\right)\left(\frac{1}{2}\ln|x^2+1|\right)+\frac{2}{25}\left(\frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right)\right)+C$$

$$=\frac{14}{25}\ln|x-2|+\frac{1}{5(x-2)}-\frac{7}{25}\ln|x^2+1|+\frac{2}{25}\tan^{-1}x+C$$

28) $\int_0^1 \frac{x}{x^2+4x+13}\,dx$

$$p = x^2+4x+13 \qquad \begin{aligned} dp &= 2x+4\,dx \\ dp &= 2(x+2)\,dx \\ \frac{1}{2}\,dp &= (x+2)\,dx \end{aligned}$$

$$\int \frac{x+2}{x^2+4x+13}\,dx = \int \frac{1}{p}\left(\frac{1}{2}\,dp\right) = \frac{1}{2}\ln|p| + C = \frac{1}{2}\ln\left|x^2+4x+13\right| + C$$

$$\begin{aligned}
\int \frac{x}{x^2+4x+13}\,dx &= \int \frac{x+2-2}{x^2+4x+13}\,dx \\
&= \int \frac{x+2}{x^2+4x+13}\,dx - \int \frac{2}{x^2+4x+13}\,dx \\
&= \int \frac{x+2}{x^2+4x+13}\,dx - \int \frac{2}{(x^2+4x+4)+9}\,dx \\
&= \int \frac{x+2}{x^2+4x+13}\,dx - \int \frac{2}{(x+2)^3+(3)^2}\,dx \\
&= \frac{1}{2}\ln\left|x^2+4x+13\right| - 2\left(\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)\right) + C \\
&= \frac{1}{2}\ln\left|x^2+4x+13\right| - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \frac{x}{x^2+4x+13}\,dx &= \left[\frac{1}{2}\ln\left|x^2+4x+13\right| - \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C\right]_0^1 \\
&= \left[\frac{1}{2}\ln\left|(1)^2+4(1)+13\right| - \frac{2}{3}\tan^{-1}\left(\frac{(1)+2}{3}\right) + C\right] \\
&\qquad - \left[\frac{1}{2}\ln\left|(0)^2+4(0)+13\right| - \frac{2}{3}\tan^{-1}\left(\frac{(0)+2}{3}\right) + C\right] \\
&= \left[\frac{1}{2}\ln|18| - \frac{2}{3}\tan^{-1}(1)\right] - \left[\frac{1}{2}\ln|13| - \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)\right] \\
&= \frac{1}{2}\ln(18) - \frac{2}{3}\left(\frac{\pi}{4}\right) - \frac{1}{2}\ln(13) + \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right) \\
&= \frac{1}{2}(\ln(18)-\ln(13)) - \frac{\pi}{6} + \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right) \\
&= \frac{1}{2}\ln\left(\frac{18}{13}\right) - \frac{\pi}{6} + \frac{2}{3}\tan^{-1}\left(\frac{2}{3}\right)
\end{aligned}$$

30) $\int \frac{x^5+x-1}{x^3+1}\,dx$

$$\begin{array}{r} x^2 \\ x^3+0x^2+0x+1 \overline{\big) x^5+0x^4+0x^3+0x^2+x-1} \\ -\underline{(x^5+0x^4+0x^3+x^2)} \\ -x^2+x-1 \end{array}$$

$$\frac{x-1}{x^3+1} = \frac{x-1}{(x+1)^1(x^2-x+1)^1} = \frac{A}{(x+1)^1} + \frac{(Bx+C)}{(x^2-x+1)^1}$$

$$x-1 = A(x^2-x+1)+(Bx+C)(x+1)$$

$$x-1 = A(x^2-x+1)+B(x^2+x)+C(x+1)$$

const term		x^2-term
$-1 = A+C$	$x-term$	$0 = A+B$
$C = (-A-1)$	$1 = -A+B+C$	$B = -A$

$$1 = -A+B+(-A-1) \qquad 2 = -2A+(-A) \qquad B = -\left(\frac{-2}{3}\right) \qquad C = -\left(\frac{-2}{3}\right)-1$$

$$2 = -2A+B \qquad 2 = -3A$$

$$A = \frac{-2}{3} \qquad B = \frac{2}{3} \qquad C = \frac{-1}{3}$$

$$\int \frac{x^5+x-1}{x^3+1}\,dx = \int \left(x^2 + \frac{(-x^2+x-1)}{x^3+1} \right) dx$$

$$= \int \left(x^2 + \frac{(-x^2)}{x^3+1} + \frac{(x-1)}{x^3+1} \right) dx$$

$$= \int \left(x^2 + \frac{(-x^2)}{x^3+1} + \frac{(\frac{-2}{3})}{(x+1)} + \frac{(\frac{2}{3})x+(\frac{-1}{3})}{(x^2-x+1)} \right) dx$$

$$= \int \left(x^2 - \frac{x^2}{x^3+1} - \frac{(\frac{2}{3})}{(x+1)} + \frac{1}{3}\left(\frac{2x-1}{x^2-x+1} \right) \right) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{3}\ln\left|x^3+1\right| - \frac{2}{3}\ln\left|x+1\right| + \frac{1}{3}\ln\left|x^2-x+1\right| + C$$

34) $\int \frac{x^4+1}{x(x^2+1)^2}\,dx$

$$\frac{x^4+1}{(x)^1(x^2+1)^2}=\frac{A}{(x)^1}+\frac{(Bx+C)}{(x^2+1)^1}+\frac{(Dx+E)}{(x^2+1)^2}$$

$$x^4+1=A(x^2+1)^2+(Bx+C)(x(x^2+1))+(Dx+E)(x)$$

$$x^4+1=A(x^4+2x^2+1)+B(x^4+x^2)+C(x^3+x)+D(x^2)+E(x)$$

	$x-term$	x^2-term		x^4-term
$const\ term$	$0=C+E$	$0=2A+B+D$	x^3-term	$1=A+B$
$1=A$	$0=(0)+E$	$0=2(1)+(0)+D$	$0=C$	$1=(1)+B$
	$E=0$	$D=-2$		$B=0$

$$\int \frac{x^4+1}{x(x^2+1)^2}\,dx=\int\left(\frac{(1)}{(x)}+\frac{(0)x+(0)}{(x^2+1)^1}+\frac{(-2)x+(0)}{(x^2+1)^2}\right)dx=\int\left(\frac{1}{(x)}-\frac{2x}{(x^2+1)^2}\right)dx$$

$$=\ln|x|-\left(\frac{-1}{x^2+1}\right)+C=\ln|x|+\frac{1}{x^2+1}+C$$

36) $\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx$

$$\begin{array}{r} 3p-3 \\ p+1\,\overline{)\,3p^2+0p+0} \\ -\underline{(3p^2+3p)} \\ -3p+0 \\ -\underline{(-3p-3)} \\ 3 \end{array}$$

$$p=\sqrt[3]{x}$$

$$p^3=x$$

$$3p^2\,dp=dx$$

$$\int \frac{1}{1+\sqrt[3]{x}}\,dx=\int\frac{1}{1+p}(3p^2\,dp)=\int\frac{3p^2}{p+1}\,dp=\int\left(3p-3+\frac{(3)}{p+1}\right)dp=\frac{3}{2}p^2-3p+3\ln|p+1|+C$$

$$=\frac{3}{2}(\sqrt[3]{x})^2-3(\sqrt[3]{x})+3\ln\left|(\sqrt[3]{x})+1\right|+C$$

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx=\left[\frac{3}{2}(\sqrt[3]{x})^2-3(\sqrt[3]{x})+3\ln\left|(\sqrt[3]{x})+1\right|+C\right]_0^1$$

$$=\left[\frac{3}{2}(\sqrt[3]{(1)})^2-3(\sqrt[3]{(1)})+3\ln\left|(\sqrt[3]{(1)})+1\right|+C\right]-\left[\frac{3}{2}(\sqrt[3]{(0)})^2-3(\sqrt[3]{(0)})+3\ln\left|(\sqrt[3]{(0)})+1\right|+C\right]$$

$$=\left[\frac{3}{2}(1)^2-3(1)+3\ln|1+1|\right]-\left[\frac{3}{2}(0)^2-3(0)+3\ln|(0)+1|\right]=3\ln 2-\frac{3}{2}$$

38) $\int_{\frac{1}{3}}^{3} \frac{\sqrt{x}}{x^2+x}\,dx$

$$p = \sqrt{x}$$
$$p^2 = x$$
$$2p\,dp = dx$$

$$\int \frac{\sqrt{x}}{x^2+x}\,dx = \int \frac{p}{(p^2)^2+p^2}(2p\,dp) = \int \frac{2p^2}{p^2(p^2+1)}\,dp = \int \frac{2}{p^2+1^2}\,dp = 2\left(\frac{1}{1}\tan^{-1}\left(\frac{p}{1}\right)\right)+C$$
$$= 2\tan^{-1} p + C = 2\tan^{-1}\sqrt{x}+C$$

$$\int_{\frac{1}{3}}^{3} \frac{\sqrt{x}}{x^2+x}\,dx = \left[2\tan^{-1}\sqrt{x}+C\right]_{\frac{1}{3}}^{3} = \left[2\tan^{-1}\sqrt{(3)}+C\right] - \left[2\tan^{-1}\sqrt{\left(\frac{1}{3}\right)}+C\right]$$
$$= 2\tan^{-1}\sqrt{3} - 2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 2\left(\frac{\pi}{3}\right) - 2\left(\frac{\pi}{6}\right) = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$$

42) $\int x\tan^{-1} x\,dx$

$$\int x\tan^{-1}x\,dx = (\tan^{-1}x)\left(\frac{1}{2}x^2\right) - \int\left(\frac{1}{2}x^2\right)\left(\frac{1}{1^2+x^2}\,dx\right)$$
$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{x^2+1}\,dx$$
$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\left(1+\frac{(-1)}{x^2+1^2}\right)dx$$
$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\left\{x+(-1)\left(\frac{1}{1}\tan^{-1}\left(\frac{x}{1}\right)\right)\right\}+C$$
$$= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \tan^{-1}x + C$$

$u_1 = \tan^{-1}x \qquad dv_1 = x\,dx$

$du_1 = \frac{1}{1^2+x^2}\,dx \qquad v_1 = \frac{1}{2}x^2$

$$\begin{array}{r} 1 \\ x^2+0x+1 \overline{)\, x^2+0x+0} \\ -(x^2+0x+1) \\ \hline -1 \end{array}$$