Definitions of Hyperbolic Functions:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \operatorname{csch} x = \frac{1}{\sinh x} \quad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \operatorname{sech} x = \frac{1}{\cosh x} \quad \coth x = \frac{\cosh x}{\sinh x}$$

Important Hyperbolic Identities:

$$\sinh(-x) = -\sinh x \quad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \quad 1 - \tanh^2 x = \operatorname{sech}^2 x \quad \coth^2 x - 1 = \operatorname{csch}^2 x$$

Derivatives of Hyperbolic Functions:

"basic 3"	"other 3"
$\frac{d}{dx}(\sinh x) = \cosh x$	$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
$\frac{d}{dx}(\cosh x) = \sinh x$	$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$	$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

For the derivative of the inverse hyperbolic functions:

1) Convert into regular hyperbolic form
2) Use the implicit differentiation
3) Convert back to original form (you must use only identities to simplify your answer)

To illustrate this procedure, see the technique below to find $\frac{d}{dx}(\tanh^{-1} x)$:

$$y = \tanh^{-1} x$$

$$\Downarrow$$

$$\tanh y = x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{(1-\tanh^2 y)} = \frac{1}{1-(x)^2} = \frac{1}{1-x^2}$$

$$\operatorname{sech}^2 y \frac{dy}{dx} = 1$$

Additional examples:

2) a) $$\tanh 0 = \frac{\sinh 0}{\cosh 0} = \frac{\frac{e^{(0)} - e^{-(0)}}{2}}{\frac{e^{(0)} + e^{-(0)}}{2}} = \frac{\frac{1-1}{2}}{\frac{1+1}{2}} = \frac{\frac{0}{2}}{\frac{2}{2}} = 0$$

b) $$\tanh 1 = \frac{\sinh 1}{\cosh 1} = \frac{\frac{e^{(1)} - e^{-(1)}}{2}}{\frac{e^{(1)} + e^{-(1)}}{2}} = \frac{\frac{e - e^{-1}}{2}}{\frac{e + e^{-1}}{2}} = \frac{e - e^{-1}}{e + e^{-1}} = \frac{e - \frac{1}{e}}{e + \frac{1}{e}} = \frac{\frac{e^2 - 1}{e}}{\frac{e^2 + 1}{e}} = \frac{e^2 - 1}{e^2 + 1}$$

4) a) $\cosh 3 = \dfrac{e^{(3)} + e^{-(3)}}{2} = \dfrac{1}{2}\left(e^3 + \dfrac{1}{e^3}\right) = \dfrac{1}{2}\left(\dfrac{(e^3)^2 + 1}{e^3}\right) = \dfrac{e^6 + 1}{2e^3}$

b) $\cosh(\ln 3) = \dfrac{e^{(\ln 3)} + e^{-(\ln 3)}}{2} = \dfrac{e^{(\ln 3)} + e^{(\ln 3^{-1})}}{2} = \dfrac{1}{2}\left(3 + 3^{-1}\right) = \dfrac{1}{2}\left(3 + \dfrac{1}{3}\right) = \dfrac{1}{2}\left(\dfrac{9}{3} + \dfrac{1}{3}\right) = \dfrac{1}{2}\left(\dfrac{10}{3}\right) = \dfrac{5}{3}$

6) a) $\sinh 1 = \dfrac{e^{(1)} - e^{-(1)}}{2} = \dfrac{1}{2}\left(e - \dfrac{1}{e}\right) = \dfrac{1}{2}\left(\dfrac{(e)^2 - 1}{e}\right) = \dfrac{e^2 - 1}{2e}$

b) We cannot use the unit circle triangle to solve this because this is an inverse hyperbolic function.

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad x \in \mathbb{R}$$

$$\sinh^{-1} 1 = \ln\left((1) + \sqrt{(1)^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right)$$

16) We can only use the identities to find the other hyperbolic functions.

$$\tanh x = \frac{12}{13}$$

$$\coth x = \frac{1}{\tanh x} = \frac{1}{\frac{12}{13}} = \frac{13}{12}$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x \quad \Rightarrow \quad \operatorname{sech} x = \sqrt{1 - \tanh^2 x}$$

$$\operatorname{sech} x = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{1 - \frac{144}{169}} = \sqrt{\frac{169}{169} - \frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\cosh x = \frac{1}{\operatorname{sech} x} = \frac{1}{\frac{5}{13}} = \frac{13}{5}$$

$$\cosh^2 x - \sinh^2 x = 1 \quad \Rightarrow \quad \sinh^2 x = \cosh^2 x - 1 \quad \Rightarrow \quad \sinh x = \sqrt{\cosh^2 x - 1}$$

$$\sinh x = \sqrt{\left(\frac{13}{5}\right)^2 - 1} = \sqrt{\frac{169}{25} - 1} = \sqrt{\frac{169}{25} - \frac{25}{25}} = \sqrt{\frac{144}{25}} = \frac{12}{5}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{1}{\frac{12}{5}} = \frac{5}{12}$$

26) $f(x) = \tanh(1 + e^{2x})$

$$\frac{df}{dx} = \operatorname{sech}^2(1 + e^{2x})(e^{2x})(2) = 2e^{2x}\operatorname{sech}^2(1 + e^{2x})$$

30) $y = x\coth(1+x^2)$

$$\frac{dy}{dx} = [1](\coth(1+x^2)) + (x)[-\operatorname{csch}^2(1+x^2)(2x)] = \coth(1+x^2) - 2x^2\operatorname{csch}^2(1+x^2)$$

32) $f(t) = \operatorname{csch} t(1-\ln \operatorname{csch} t)$

$$\frac{df}{dt} = [-\operatorname{csch} t \coth t](1-\ln\operatorname{csch} t) + (\operatorname{csch} t)\left[\frac{1}{\operatorname{csch} t}(-\operatorname{csch} t\coth t)\right]$$

$$= -\operatorname{csch} t\coth t\{(1-\ln\operatorname{csch} t)+1\} = -\operatorname{csch} t\coth t\{2-\ln\operatorname{csch} t\}$$

34) $y = \sinh(\cosh x)$

$$\frac{dy}{dx} = \cosh(\cosh x)(\sinh x) = \sinh x\cosh(\cosh x)$$

40) $y = \operatorname{sech}^{-1}(e^{-x})$

$$y = \operatorname{sech}^{-1}(e^{-x})$$

$$\operatorname{sech} y = e^{-x}$$

$$-\operatorname{sech} y\tanh y\frac{dy}{dx} = e^{-x}(-1)$$

$$\frac{dy}{dx} = \frac{e^{-x}}{\operatorname{sech} y\tanh y}$$

$$= \frac{e^{-x}}{(e^{-x})\sqrt{1-(e^{-x})^2}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-e^{-2x}}}$$

$$\cosh^2 y - \sinh^2 y = 1$$

$$1-\tanh^2 y = \operatorname{sech}^2 y$$

$$\tanh^2 y = 1-\operatorname{sech}^2 y$$

$$\tanh y = \pm\sqrt{1-\operatorname{sech}^2 y}$$

$$\tanh y = \sqrt{1-(e^{-x})^2}$$

54) $\int \sinh(1+4x)\,dx$

$$p = 1+4x \quad dp = 4\,dx \Rightarrow \frac{1}{4}\,dp = dx$$

$$\int \sinh(1+4x)\,dx = \int \sinh p\left(\frac{1}{4}\,dp\right)$$

$$= \frac{1}{4}\cosh p + C$$

$$= \frac{1}{4}\cosh(1+4x) + C$$

56) $\int \tanh x \, dx$

$p = \cosh x \quad dp = \sinh x \, dx$

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$$
$$= \int \frac{1}{p} \, dp$$
$$= \ln|p| + C$$
$$= \ln|\cosh x| + C$$

58) $\int \frac{\operatorname{sech}^2 x}{2 + \tanh x} \, dx$

$p = 2 + \tanh x \quad dp = \operatorname{sech}^2 x \, dx$

$$\int \frac{\operatorname{sech}^2 x}{2 + \tanh x} \, dx = \int \frac{1}{p} \, dp$$
$$= \ln|p| + C$$
$$= \ln|2 + \tanh x| + C$$