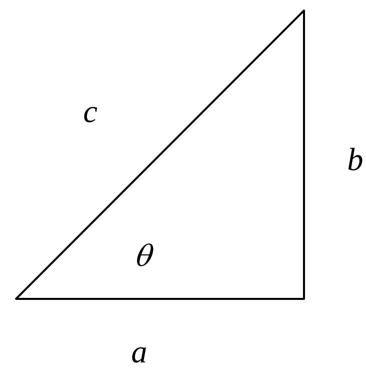
Read the section for the restrictions applied to the trigonometric functions so they become 1-1and inverse function exists.

The integration formulas are obtainable using section 6.2, so we don't emphasize in this section.

Useful information from precalculus class:

$$c=\sqrt{a^2+b^2} \quad \sin\theta=\frac{b}{c} \quad \csc\theta=\frac{c}{b}$$
$$b=\sqrt{c^2-a^2} \quad \cos\theta=\frac{a}{c} \quad \sec\theta=\frac{c}{a}$$
$$a=\sqrt{c^2-b^2} \quad \tan\theta=\frac{b}{a} \quad \cot\theta=\frac{a}{b}$$




$$\cos^2\theta+\sin^2\theta=1 \quad 1+\tan^2\theta=\sec^2\theta \quad \cot^2\theta+1=\csc^2\theta$$

Given the restrictions of the trigonometric functions are met, then:

$$\theta=\sin^{-1}k=\arcsin k \iff \sin\theta=k \qquad \theta=\csc^{-1}k=\operatorname{arc}\csc k \iff \csc\theta=k$$
$$\theta=\cos^{-1}k=\arccos k \iff \cos\theta=k \qquad \theta=\sec^{-1}k=\operatorname{arc}\sec k \iff \sec\theta=k$$
$$\theta=\tan^{-1}k=\arctan k \iff \tan\theta=k \qquad \theta=\cot^{-1}k=\operatorname{arc}\cot k \iff \cot\theta=k$$

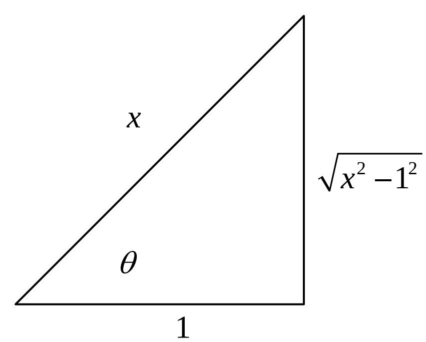
For the derivative of the inverse trigonometric functions:

1) Convert into regular trigonometric form
2) Use the implicit differentiation
3) Convert back to original form (you need to draw the appropriate triangle to simplify your answer)

To illustrate this, the solution for the question 14 is shown first.

14) Prove that $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$

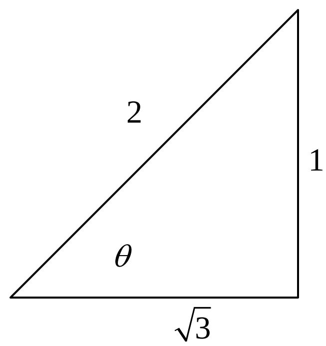
$$\theta=\sec^{-1}x$$
$$\Downarrow$$
$$\sec\theta=x$$
$$\sec\theta\tan\theta\frac{d\theta}{dx}=1$$
$$\frac{d\theta}{dx}=\frac{1}{\sec\theta\tan\theta}$$
$$=\frac{1}{\left(\frac{x}{1}\right)\left(\frac{\sqrt{x^2-1^2}}{1}\right)}$$
$$=\frac{1}{x\sqrt{x^2-1}}$$



Additional examples:

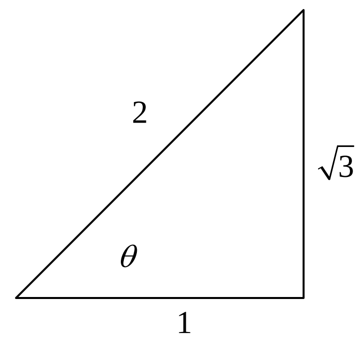
2) a) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$\Downarrow$$
$$\tan\theta = \frac{1}{\sqrt{3}}$$
$$\theta = \frac{\pi}{6}$$
$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$



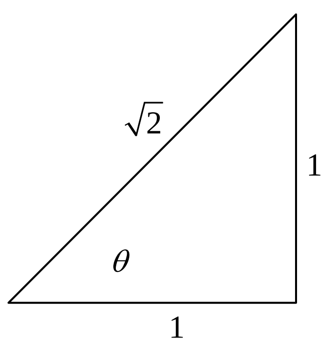

b) $\sec^{-1} 2$

$$\theta = \sec^{-1} 2$$
$$\Downarrow$$
$$\sec\theta = 2 = \frac{2}{1}$$
$$\theta = \frac{\pi}{3}$$
$$\sec^{-1} 2 = \frac{\pi}{3}$$



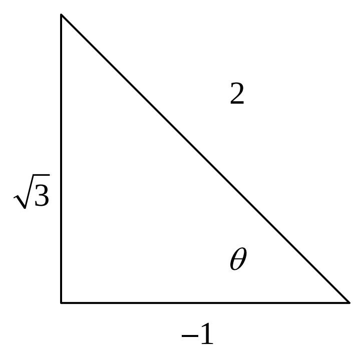

4) a) $\arctan 1$

$$\theta = \tan^{-1} 1$$
$$\Downarrow$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$
$$\arctan 1 = \frac{\pi}{4}$$



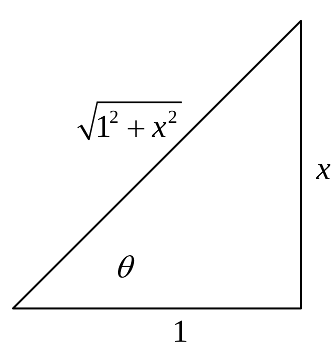

b) $\arccos\left(\frac{-1}{2}\right)$

$$\theta = \cos^{-1}\left(\frac{-1}{2}\right)$$
$$\Downarrow$$
$$\cos\theta = \frac{-1}{2}$$
$$\theta = \frac{2\pi}{3}$$
$$\arccos\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$$



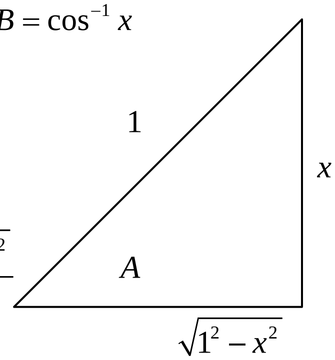

10) $\cos(2\tan^{-1} x)$

$$\cos(2\underbrace{\tan^{-1} x}_{\theta})$$
$$\theta = \tan^{-1} x$$
$$\Downarrow$$
$$\tan\theta = x = \frac{x}{1}$$
$$\cos(2\tan^{-1} x) = \cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= \left(\frac{1}{\sqrt{1^2+x^2}}\right)^2 - \left(\frac{x}{\sqrt{1^2+x^2}}\right)^2$$
$$= \frac{1^2 - x^2}{1^2 + x^2} = \frac{1-x^2}{1+x^2}$$

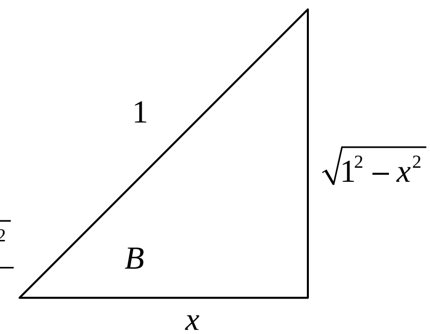


12) a) let $A = \sin^{-1} x$ and $B = \cos^{-1} x$

$$A = \sin^{-1} x$$
$$\Downarrow$$
$$\sin A = x = \frac{x}{1}$$
$$\cos A = \frac{\sqrt{1^2 - x^2}}{1}$$




$$B = \cos^{-1} x$$
$$\Downarrow$$
$$\cos B = x = \frac{x}{1}$$
$$\sin B = \frac{\sqrt{1^2 - x^2}}{1}$$




now

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

use each side as an angle and apply to sine function

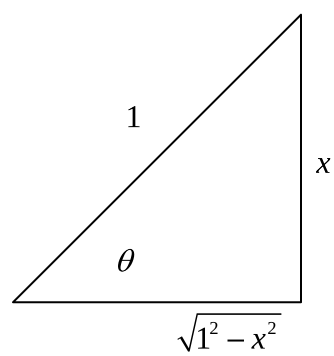
$$A + B = \frac{\pi}{2}$$
$$\sin(A + B) = \sin\left(\frac{\pi}{2}\right) = 1$$

We just have to show that $\sin(A + B) = 1$, using the addition formula and the triangles above,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B = \left(\frac{x}{1}\right)\left(\frac{x}{1}\right) + \left(\frac{\sqrt{1^2 - x^2}}{1}\right)\left(\frac{\sqrt{1^2 - x^2}}{1}\right) = x^2 + (1^2 - x^2) = 1$$

b) First, we must find the derivative of $\sin^{-1} x$

$$\theta = \sin^{-1} x$$
$$\Downarrow$$
$$\sin\theta = x$$
$$\cos\theta \frac{d\theta}{dx} = 1$$
$$\frac{d\theta}{dx} = \frac{1}{\cos\theta} = \frac{1}{\left(\frac{\sqrt{1^2 - x^2}}{1}\right)} = \frac{1}{\sqrt{1 - x^2}}$$




Using implicit differentiation with respect to *x*,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$
$$\frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2}\right)$$
$$\frac{1}{\sqrt{1 - x^2}} + \frac{d}{dx}(\cos^{-1} x) = 0$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1 - x^2}}$$

20) $y = \tan^{-1}\left(x - \sqrt{1+x^2}\right)$

$$\tan y = x - \sqrt{1+x^2}$$

$$\sec^2 y \frac{dy}{dx} = 1 - \frac{x}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \left\{1 - \frac{x}{\sqrt{1+x^2}}\right\}\frac{1}{\sec^2 y}$$

$$= \left\{1 - \frac{x}{\sqrt{1+x^2}}\right\}\frac{1}{\left(\sqrt{1^2 + \left(x - \sqrt{1+x^2}\right)^2}\right)^2}$$

$$= \left\{\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}\right\}\frac{1}{1 + (x^2 - 2x\sqrt{1+x^2} + (1+x^2))}$$

$$= \left\{\frac{\sqrt{1+x^2} - x}{\sqrt{1+x^2}}\right\}\frac{1}{2 + 2x^2 - 2x\sqrt{1+x^2}}$$

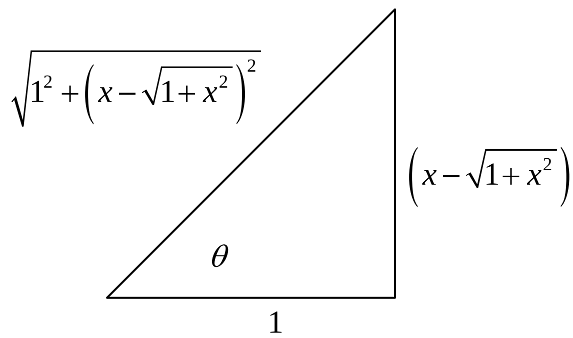
$$= \left\{\frac{\sqrt{1+x^2} - x}{\sqrt{1+x^2}}\right\}\frac{1}{2(1+x^2) - 2x\sqrt{1+x^2}}$$

$$= \left\{\frac{\sqrt{1+x^2} - x}{\sqrt{1+x^2}}\right\}\frac{1}{2\left(\sqrt{1+x^2}\right)^2 - 2x\sqrt{1+x^2}}$$

$$= \left\{\frac{\sqrt{1+x^2} - x}{\sqrt{1+x^2}}\right\}\frac{1}{2\sqrt{1+x^2}\left(\sqrt{1+x^2} - x\right)}$$

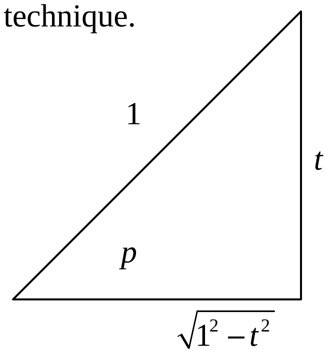
$$= \frac{\sqrt{1+x^2} - x}{2\left(\sqrt{1+x^2}\right)^2\left(\sqrt{1+x^2} - x\right)}$$

$$= \frac{1}{2(1+x^2)}$$

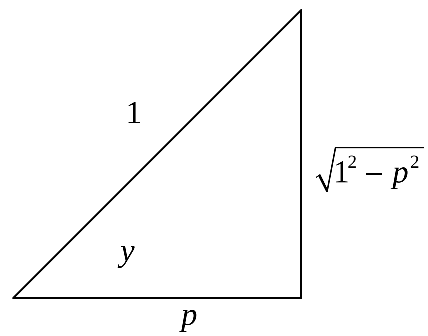


24) $y = \cos^{-1}(\sin^{-1} t)$

We need to split this into 2 parts and use the chain rule technique.




$$p = \sin^{-1} t$$
$$\Downarrow$$
$$\sin p = t$$
$$\cos p \frac{dp}{dt} = 1$$
$$\frac{dp}{dt} = \frac{1}{\cos p} = \frac{1}{\left(\frac{\sqrt{1^2 - t^2}}{1}\right)} = \frac{1}{\sqrt{1-t^2}}$$

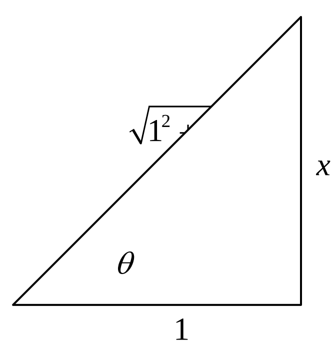



$$y = \cos^{-1} p$$
$$\Downarrow$$
$$\cos y = p$$
$$-\sin y \frac{dy}{dp} = 1$$
$$\frac{dy}{dp} = \frac{-1}{\sin y} = \frac{-1}{\left(\frac{\sqrt{1^2 - p^2}}{1}\right)} = \frac{-1}{\sqrt{1-p^2}}$$

Therefore,

$$\frac{dy}{dt} = \left(\frac{dy}{dp}\right)\left(\frac{dp}{dt}\right) = \left(\frac{-1}{\sqrt{1-p^2}}\right)\left(\frac{dp}{dt}\right) = \left(\frac{-1}{\sqrt{1-(\sin^{-1} t)^2}}\right)\left(\frac{1}{\sqrt{1-t^2}}\right) = \frac{-1}{\sqrt{1-t^2}\,\sqrt{1-(\sin^{-1} t)^2}}$$

26) $f(x) = x \ln(\arctan x)$




$$\theta = \tan^{-1} x$$
$$\Downarrow$$
$$\tan\theta = x$$
$$\sec^2\theta \frac{d\theta}{dx} = 1$$
$$\frac{d\theta}{dx} = \frac{1}{\sec^2\theta} = \frac{1}{\left(\frac{\sqrt{1^2 + x^2}}{1}\right)^2} = \frac{1}{1^2 + x^2}$$

$$\frac{df}{dx} = [1](\ln(\tan^{-1} x)) + (x)\left[\frac{1}{\tan^{-1} x}\left(\frac{1}{1^2 + x^2}\right)\right]$$
$$= \ln(\tan^{-1} x) + \frac{x}{(1+x^2)\tan^{-1} x}$$

28) $y = \arctan\sqrt{\frac{1-x}{1+x}}$

$$y = \arctan\sqrt{\frac{1-x}{1+x}}$$

$$y = \tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

$$\tan y = \sqrt{\frac{1-x}{1+x}}$$

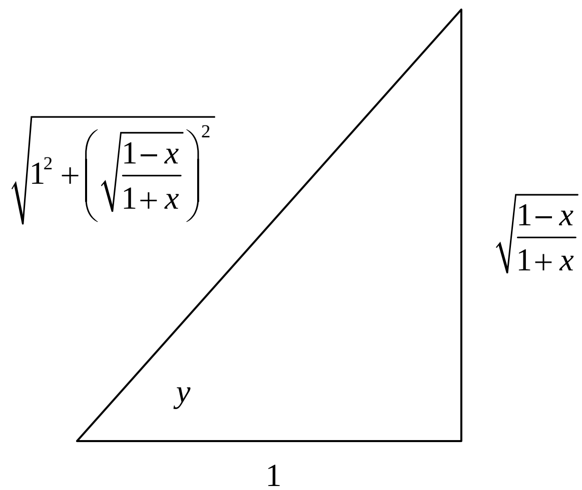
$$\tan^2 y = \frac{1-x}{1+x}$$

$$2\tan y \sec^2 y \frac{dy}{dx} = \frac{[-1](1+x)-(1-x)[1]}{(1+x)^2}$$

$$2\tan y \sec^2 y \frac{dy}{dx} = \frac{-2}{(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-2}{2\tan y \sec^2 y(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{\tan y \sec^2 y(1+x)^2}$$




$$\frac{dy}{dx} = \frac{-1}{\left(\sqrt{\frac{1-x}{1+x}}\right)\left(\sqrt{1^2+\left(\sqrt{\frac{1-x}{1+x}}\right)^2}\right)^2(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{\left(\sqrt{\frac{1-x}{1+x}}\right)\left(1+\frac{1-x}{1+x}\right)(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{\left(\sqrt{\frac{1-x}{1+x}}\right)\left(\frac{1+x}{1+x}+\frac{1-x}{1+x}\right)(1+x)^2}$$

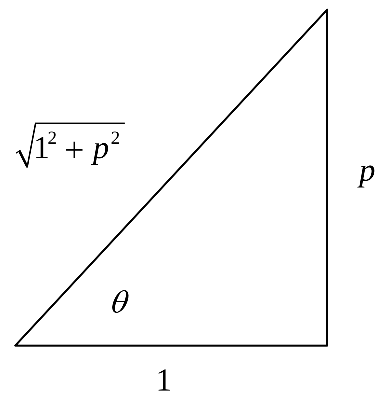
$$\frac{dy}{dx} = \frac{-1}{\left(\sqrt{\frac{1-x}{1+x}}\right)\left(\frac{2}{1+x}\right)(1+x)^2}$$

$$\frac{dy}{dx} = \frac{-1}{2(1+x)\left(\sqrt{\frac{1-x}{1+x}}\right)} = \frac{-1}{2(\sqrt{1+x})^2\left(\frac{\sqrt{1-x}}{\sqrt{1+x}}\right)}$$

$$\frac{dy}{dx} = \frac{-1}{2(\sqrt{1+x})(\sqrt{1-x})} = \frac{-1}{2\sqrt{(1+x)(1-x)}} = \frac{-1}{2\sqrt{1-x^2}}$$

30) $\tan^{-1}(xy) = 1 + x^2 y \qquad \frac{dy}{dx} = ?$

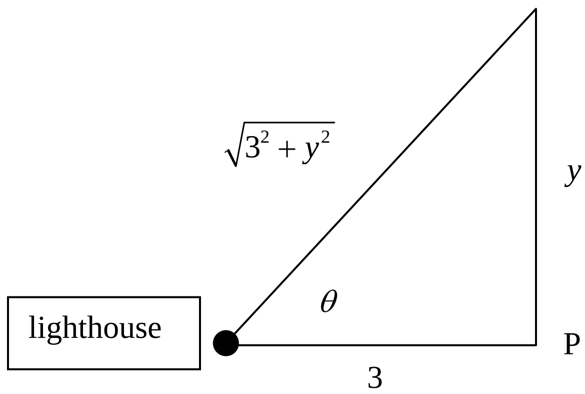
$$\theta = \tan^{-1} p$$
$$\tan\theta = p$$
$$\sec^2\theta \frac{d\theta}{dp} = 1$$
$$\frac{d\theta}{dp} = \frac{1}{\sec^2\theta}$$
$$\frac{d\theta}{dp} = \frac{1}{(\sqrt{1^2 + p^2})^2}$$
$$\frac{d\theta}{dp} = \frac{1}{(1 + p^2)}$$




let $p = xy$

$$\tan^{-1}(xy) = 1 + x^2 y$$
$$\frac{1}{(1+(xy)^2)}\left\{[1](y) + (x)\left[1\frac{dy}{dx}\right]\right\} = 0 + \left\{[2x](y) + (x^2)\left[1\frac{dy}{dx}\right]\right\}$$
$$\frac{y}{(1+x^2y^2)} + \frac{x}{(1+x^2y^2)}\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx}$$
$$\frac{x}{(1+x^2y^2)}\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy - \frac{y}{(1+x^2y^2)}$$
$$\frac{dy}{dx}\left(\frac{x}{(1+x^2y^2)} - x^2\right) = 2xy - \frac{y}{(1+x^2y^2)}$$
$$\frac{dy}{dx} = \frac{2xy - \frac{y}{(1+x^2y^2)}}{\frac{x}{(1+x^2y^2)} - x^2}$$
$$\frac{dy}{dx} = \frac{2xy(1+x^2y^2) - y}{x - x^2(1+x^2y^2)}$$

36) This is a related rates problem from Calculus 1.
Diagram of the problem is given to the right.




$$\frac{d\theta}{dt} = 4 \text{ rev/min} = (4 \text{ rev/min})(2\pi \text{ rad/rev})$$
$$= 8\pi \text{ rad/min} = (8\pi \text{ rad/min})(60 \text{ min/hr})$$
$$= 480\pi \text{ rad/hr}$$

When $y = 1\,\text{km}$, $\frac{dy}{dt} = ?$

$$\tan\theta = \frac{y}{3}$$
$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{3}\frac{dy}{dt}$$
$$\frac{dy}{dt} = 3\sec^2\theta \frac{d\theta}{dt}$$
$$\frac{dy}{dt} = 3\left(\frac{\sqrt{3^2+y^2}}{3}\right)^2 \frac{d\theta}{dt}$$

Since the rate of angle changing is constant, we get

$$\left.\frac{dy}{dt}\right|_{y=1} = 3\left(\frac{\sqrt{3^2+(1)^2}}{3}\right)^2 (480\pi) = \frac{10}{3}(480\pi) = 1600\pi \text{ km/hr}$$