Exponential Growth and Decay (basic form): The differential equation $\frac{dP}{dt} = rP$ with a condition $P(0) = P_0$ yields a solution:
Exponential growth (Relative Growth) or Radioactive Decay Model:

$$P(t) = P_0 e^{rt}$$

For Exponential Growth	For Radioactive Decay
$P(t) =$ population at time t	$P(t) =$ amount of mass left at time t
$P_0 =$ initial size of the population	$P_0 =$ initial mass
$r =$ relative rate of growth (expressed as proportion of population)	$r =$ decay constant (the value is negative or needs to be computed)
$t =$ time	$t =$ time

Doubling time $P(t) = 2P_0$: amount of time needed for population to double its initial size

Half-life $P(t) = \frac{1}{2} P_0$: amount of time needed for the radioactive material reduce (decay) and have half left.

The actual solving technique is called separation of variables shown below:
Given $\frac{dP}{dt} = rP$ with a condition $P(0) = P_0$.

$$\frac{dP}{dt} = rP$$

$$\frac{1}{P}\,dP = r\,dt$$

$$\int \frac{1}{P}\,dP = \int r\,dt \quad \Rightarrow \quad \begin{aligned} P(0) = P_0 &= C_2 e^{r(0)} \\ P_0 &= C_2 \end{aligned} \quad \Rightarrow \quad P(t) = P_0 e^{rt}$$

$$\ln|P| = rt + c_1$$

$$P = e^{(rt + c_1)} = (e^{rt})(e^{c_1})$$

$$P = C_2 e^{rt}$$

For Newton's Law of Cooling, we just need to modify the differential equation given above. Usually in this case, it is a function of t with, T_s, surrounding temperature that we consider to be constant.

Instead of our differential equation having a simple variable P, we replace with $(T - T_s)$ to get $\frac{dT}{dt} = r(T - T_s)$ and we can use the method of separation of variables to find a solution of $T(t)$.

To illustrate, solution of exercise 14 is shown first below:

14) normal body temperature $= 37.0°C$ $\quad T_s = 20.0°C$

Time	t in minutes	Temperature	$T(t)$
1:30PM	0	$32.5°C$	$T(0) = 32.5$
2:30PM	60	$30.3°C$	$T(60) = 30.3$

We first solve the differential equation with condition $T(0) = 32.5$.

$$\frac{dT}{dt} = r(T-20)$$

$$\frac{1}{(T-20)}\,dT = r\,dt$$

$$\int \frac{1}{(T-20)}\,dT = \int r\,dt$$

$$\ln\left|(T-20)\right| = rt + c_1$$

$$(T-20_s) = e^{(rt+c_1)} = (e^{rt})(e^{c_1})$$

$$(T-20) = C_2 e^{rt}$$

$$\Rightarrow \quad \begin{aligned} T(0) &= 32.5 \\ ((32.5)-20) &= C_2 e^{r(0)} \\ 12.5 &= C_2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} (T-20) &= 12.5e^{rt} \\ T(t) &= 20 + 12.5e^{rt} \end{aligned}$$

Now apply the second condition, $T(60) = 30.3$, to the solution above to find *r*.

$$30.3 = T(60) = 20 + 12.5e^{r(60)}$$

$$10.3 = 12.5e^{60r}$$

$$\frac{10.3}{12.5} = e^{60r} \qquad \Rightarrow \quad T(t) = 20 + 12.5e^{\left(\frac{1}{60}\ln\left(\frac{10.3}{12.5}\right)\right)t}$$

$$\ln\left(\frac{10.3}{12.5}\right) = 60r$$

$$r = \frac{1}{60}\ln\left(\frac{10.3}{12.5}\right)$$

We now have the specific function of our body. Now use the normal body temperature to find our how long ago murder has taken place. $T(t) = 37.0°C \quad t = ?$

$$37.0 = 20 + 12.5e^{\left(\frac{1}{60}\ln\left(\frac{10.3}{12.5}\right)\right)t}$$

$$17 = 12.5e^{\left(\frac{1}{60}\ln\left(\frac{10.3}{12.5}\right)\right)t}$$

$$\frac{17}{12.5} = e^{\left(\frac{1}{60}\ln\left(\frac{10.3}{12.5}\right)\right)t}$$

$$\ln\left(\frac{17}{12.5}\right) = \left(\frac{1}{60}\ln\left(\frac{10.3}{12.5}\right)\right)t$$

$$t = \frac{60\ln\left(\frac{17}{12.5}\right)}{\ln\left(\frac{10.3}{12.5}\right)}$$

For the examination, your final answer must look like the answer above because calculators are not allowed. If this was a Physics course or calculators would be allowed, then $t \approx -95.302$ minutes. Rounding off the values to the nearest minute, the murder took place 95 minutes before 3:30PM or 11:55AM.

Additional examples:

2) $P(t) = P_0 e^{rt}$

t in hours	Ratio	$P(t)$	
0	1	60	$P(0) = 60 = P_0$
$\frac{1}{3}$	2	120	$P(\frac{1}{3}) = 120$

a) $120 = 60e^{r\left(\frac{1}{3}\right)}$, $2 = e^{\frac{1}{3}r}$ $\Rightarrow$ $\ln 2 = \frac{1}{3}r$ $\Rightarrow$ $r = 3\ln 2 = \ln(2^3) = \ln 8$

b) $P(t) = 60e^{(\ln 8)t} = 60e^{\ln(8^t)} = 60(8^t)$

c) $P(8) = 60(8^{(8)})$

d) Using the differential equation, $\frac{dP}{dt} = rP = (\ln 8)(60(8^{(8)}))$

$P(t) = 20000 \quad t = ?$

e) $20000 = 60(8^t)$, $\frac{20000}{60} = 8^t$ $\Rightarrow$ $\frac{1000}{3} = 8^t$ $\Rightarrow$ $t = \log_8\left(\frac{1000}{3}\right) = \frac{\ln\left(\frac{1000}{3}\right)}{\ln 8}$

6) $P(t) = P_0 e^{rt}$

year	1951	1961	1981	2001	2010	2020
t in year	0	10	30	50	60	70
$P(t)$	361	439	683	Calculate	Calculate	Calculate
	$P(0) = 361 = P_0$	$P(10) = 439$	$P(30) = 683$	$P(50) = ?$	$P(60) = ?$	$P(70) = ?$

a) $439 = 361e^{r(10)}$, $\frac{439}{361} = e^{10r}$, $\ln\left(\frac{439}{361}\right) = 10r$ $\Rightarrow$ $r = \frac{1}{10}\ln\left(\frac{439}{361}\right)$, $P(t) = 361e^{\left(\frac{1}{10}\ln\left(\frac{439}{361}\right)\right)t}$ $\Rightarrow$

$$P(50) = 361e^{\left(\frac{1}{10}\ln\left(\frac{439}{361}\right)\right)(50)} = 361e^{5\ln\left(\frac{439}{361}\right)} = 361e^{\ln\left(\frac{439}{361}\right)^5} = 361\left(\frac{439}{361}\right)^5$$

$683 = 361e^{r(30)}$, $\frac{683}{361} = e^{30r}$, $\ln\left(\frac{683}{361}\right) = 30r$ $\Rightarrow$ $r = \frac{1}{30}\ln\left(\frac{683}{361}\right)$, $P(t) = 361e^{\left(\frac{1}{30}\ln\left(\frac{683}{361}\right)\right)t}$ $\Rightarrow$

$$P(50) = 361e^{\left(\frac{1}{30}\ln\left(\frac{683}{361}\right)\right)50} = 361e^{\frac{5}{3}\ln\left(\frac{683}{361}\right)} = 361e^{\ln\left(\frac{683}{361}\right)^{5/3}} = 361\left(\frac{683}{361}\right)^{5/3} = 361\left(\sqrt[3]{\frac{683}{361}}\right)^5$$

b)

$$P(60) = 361e^{\left(\frac{1}{30}\ln\left(\frac{683}{361}\right)\right)60} = 361e^{2\ln\left(\frac{683}{361}\right)} = 361e^{\ln\left(\frac{683}{361}\right)^2} = 361\left(\frac{683}{361}\right)^2$$

$$P(70) = 361e^{\left(\frac{1}{30}\ln\left(\frac{683}{361}\right)\right)70} = 361e^{\frac{7}{3}\ln\left(\frac{683}{361}\right)} = 361e^{\ln\left(\frac{683}{361}\right)^{7/3}} = 361\left(\frac{683}{361}\right)^{7/3} = 361\left(\sqrt[3]{\frac{683}{361}}\right)^7$$

8) $P(t) = P_0 e^{rt}$; half-life 28 days: $P(28) = \frac{1}{2} P_0$

$$\frac{1}{2}P_0 = P_0 e^{r(28)} \quad \frac{1}{2} = e^{28r} \quad \Rightarrow \quad \ln\left(\frac{1}{2}\right) = 28r \quad r = \frac{1}{28}\ln\left(\frac{1}{2}\right) \quad \Rightarrow \quad P(t) = P_0 e^{\left(\frac{1}{28}\ln\left(\frac{1}{2}\right)\right)t}$$

a) $P(0) = P_0 = 50$ mg $\Rightarrow$ $P(t) = 50e^{\left(\frac{1}{28}\ln\left(\frac{1}{2}\right)\right)t}$

b) $P(40) = 50e^{\left(\frac{1}{28}\ln\left(\frac{1}{2}\right)\right)(40)} = 50e^{\left(\frac{40}{28}\ln\left(\frac{1}{2}\right)\right)} = 50e^{\left(\frac{10}{7}\ln\left(\frac{1}{2}\right)\right)} = 50e^{\left(\ln\left(\frac{1}{2}\right)^{10/7}\right)} = 50\left(\frac{1}{2}\right)^{10/7} = \frac{50}{2^{10/7}} = \frac{50}{\left(\sqrt[7]{2}\right)^{10}}$

c) $P(t) = 2$ mg, $t = ?$

$$2 = 50e^{\left(\frac{1}{28}\ln\left(\frac{1}{2}\right)\right)t} \quad \frac{2}{50} = e^{\left(\frac{1}{28}\ln\left(\frac{1}{2}\right)\right)t} \quad \Rightarrow \quad \ln\left(\frac{1}{25}\right) = \left(\frac{1}{28}\ln\left(\frac{1}{2}\right)\right)t \quad \Rightarrow \quad t = \frac{28\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{1}{2}\right)}$$

d) Omit

12) Since the differential equation is the 1st derivative of a special equation, we get $\frac{dP}{dt} = 2P$. So we can extract the value of $r = 2$. Also, $(0,5):(t,P) \Rightarrow P(0) = 5 = P_0$

$P(t) = P_0 e^{rt} \quad \Rightarrow \quad P(t) = 5e^{2t}$

Now replacing *t* by *x* and $P(t)$ by *y*, we get $y = 5e^{2x}$.

16) $T_s = 20°C$, $\frac{dT}{dt} = -1°C/\min$ when $T(t) = 70°C$, $t = ?$ $\qquad T(0) = 95°C$

$$\frac{dT}{dt} = r(T - T_s) \qquad -1 = r(70-20) \quad \Rightarrow \quad r = \frac{-1}{50}$$

$$\frac{dT}{dt} = \frac{-1}{50}(T-20)$$

$$\frac{1}{(T-20)}\,dT = \frac{-1}{50}\,dt$$

$$\int \frac{1}{(T-20)}\,dT = \int \frac{-1}{50}\,dt$$

$$\ln\left|(T-20)\right| = \frac{-1}{50}t + c_1$$

$$(T-20_s) = e^{(\frac{-1}{50}t + c_1)} = (e^{\frac{-1}{50}t})(e^{c_1})$$

$$(T-20) = C_2 e^{\frac{-1}{50}t}$$

$\Rightarrow$

$$T(0) = 95$$

$$((95)-20) = C_2 e^{\frac{-1}{50}(0)}$$

$$75 = C_2$$

$$(T-20) = 75e^{\frac{-1}{50}t}$$

$$T(t) = 20 + 75e^{\frac{-1}{50}t}$$

$\Rightarrow$

$$70 = 20 + 75e^{\frac{-1}{50}t}$$

$$50 = 75e^{\frac{-1}{50}t}$$

$$\frac{50}{75} = e^{\frac{-1}{50}t}$$

$$\ln\left(\frac{2}{3}\right) = \frac{-1}{50}t$$

$$t = -50\ln\left(\frac{2}{3}\right)$$