Partial Derivatives of a Function of Two Variables

Definition

The **partial derivative of** $f(x,y)$ **with respect to** x **at the point** (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \frac{d}{dx} f(x, y_0)\Big|_{x=x_0} = \lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

Definition

The **partial derivative of** $f(x,y)$ **with respect to** y **at the point** (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \frac{d}{dy} f(x_0, y)\Big|_{y=y_0} = \lim_{h\to 0}\frac{f(x_0, y_0+h) - f(x_0, y_0)}{h},$$

provided the limit exists.

Second-Order Partial Derivatives

$\frac{\partial^2 f}{\partial x^2} = f_{xx}$ Differentiate with respect to x twice

$\frac{\partial^2 f}{\partial y^2} = f_{yy}$ Differentiate with respect to y twice

$\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ Differentiate first with respect to y, then with respect to x

$\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$ Differentiate first with respect to x, then with respect to y

The Mixed Derivative Theorem

Theorem 2 - The Mixed Derivative Theorem

If $f(x,y)$ and its partial derivatives $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f}{\partial y} = f_y$, $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$, and $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ are defined throughout an open interval region containing a point (a,b) and are all continuous at (a,b), then

$$\frac{\partial^2}{\partial y \partial x}\big(f(a,b)\big) = \frac{\partial^2}{\partial x \partial y}\big(f(a,b)\big) \quad \text{or} \quad f_{xy}(a,b) = f_{yx}(a,b)$$

Partial Derivatives of Still Higher Order

$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$ Differentiate first with respect to y twice, then with respect to x

$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx}$ Differentiate first with respect to y twice, then with respect to x twice

Differentiability

Definition

A function $f(x,y)$ is **differentiable at** (x_0, y_0) if $\frac{\partial}{\partial x}\left(f(x_0,y_0)\right) = f_x(x_0,y_0)$ and $\frac{\partial}{\partial y}\left(f(x_0,y_0)\right) = f_y(x_0,y_0)$ exist and $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$ satisfies an equation of the form

$$\Delta z = f(x_0,y_0)\Delta x + f(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

Theorem 3 – The Increment Theorem for Functions of Two Variables

Suppose that the first partial derivatives of $f(x,y)$ are defined throughout an open region R containing the point (x_0,y_0) and that $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$ are continuous at (x_0,y_0). Then the change

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$

in the value of f that results from moving from (x_0,y_0) to another point $(x_0+\Delta x, y_0+\Delta y)$ in R satisfies an equation of the form

$$\Delta z = f(x_0,y_0)\Delta x + f(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

in which each of $\varepsilon_1, \varepsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

Corollary of Theorem 3

If the partial derivatives $\frac{\partial f}{\partial x} = f_x$ and $\frac{\partial f}{\partial y} = f_y$ of a function $f(x,y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

Theorem 4 – Differentiability Implies Continuity

If a function $f(x,y)$ is differentiable at (x_0,y_0), then f is continuous at (x_0,y_0).

2) $f(x,y) = x^2 - xy + y^2$

$$\frac{\partial f}{\partial x} = [2x] - y[1] + 0 = 2x - y$$

$$\frac{\partial f}{\partial y} = 0 - x[1] + [2y] = -x + 2y$$

---

4) $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

$$\frac{\partial f}{\partial x} = 5y[1] - 7[2x] - 0 + 3[1] - 0 + 0 = 5y - 14x + 3$$

$$\frac{\partial f}{\partial y} = 5x[1] - 0 - [2y] + 0 - 6[1] + 2 = 5x - 2y - 6$$

---

6) $f(x,y) = (2x - 3y)^3$

$$\frac{\partial f}{\partial x} = 3(2x - 3y)^2(2) = 6(2x - 3y)^2$$

$$\frac{\partial f}{\partial y} = 3(2x - 3y)^2(-3) = -9(2x - 3y)^2$$

---

8) $f(x,y) = \left(x^3 + \frac{y}{2}\right)^{2/3}$

$$\frac{\partial f}{\partial x} = \frac{2}{3}\left(x^3 + \frac{y}{2}\right)^{\frac{-1}{3}}(3x^2) = \frac{2x^2}{\sqrt[3]{x^3 + \frac{y}{2}}}$$

$$\frac{\partial f}{\partial y} = \frac{2}{3}\left(x^3 + \frac{y}{2}\right)^{\frac{-1}{3}}\left(\frac{1}{2}\right) = \frac{1}{3\sqrt[3]{x^3 + \frac{y}{2}}}$$

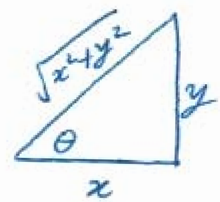
10) $f(x,y) = \dfrac{x}{x^2+y^2} = x\,(x^2+y^2)^{-1}$

$$\frac{\partial f}{\partial x} = \frac{(x^2+y^2)[1]-(x)[2x]}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = x\left[-1(x^2+y^2)^{-2}(2y)\right] = \frac{-2xy}{(x^2+y^2)^2}$$

---

12) $f(x,y) = \tan^{-1}\left(\dfrac{y}{x}\right)$



$\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$\Downarrow$

$\tan\theta = \dfrac{y}{x} = yx^{-1}$

$\sec^2\theta\,\dfrac{d\theta}{dx} = y[-1x^{-2}]$

$$\frac{d\theta}{dx} = \frac{-y}{x^2\sec^2\theta}$$

$$= \frac{-y}{x^2\left(\frac{\sqrt{x^2+y^2}}{x}\right)^2} = \frac{-y}{x^2+y^2} \Rightarrow \frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2}$$

using same set up as above

$\sec^2\theta\,\dfrac{d\theta}{dy} = \dfrac{1}{x}[1]$

$$\frac{d\theta}{dy} = \frac{1}{x\sec^2\theta}$$

$$= \frac{1}{x\left(\frac{\sqrt{x^2+y^2}}{x}\right)^2} = \frac{x}{x^2+y^2} \Rightarrow \frac{\partial f}{\partial y} = \frac{x}{x^2+y^2}$$

---

14) $f(x,y) = e^{-x}\sin(x+y)$

$$\frac{\partial f}{\partial x} = (e^{-x})\left[\cos(x+y)(1)\right] + (\sin(x+y))\left[e^{-x}(-1)\right]$$

$$= e^{-x}\cos(x+y) - e^{-x}\sin(x+y) = \frac{\cos(x+y)}{e^x} - \frac{\sin(x+y)}{e^x}$$

$$= \frac{\cos(x+y)-\sin(x+y)}{e^x}$$

$$\frac{\partial f}{\partial x} = e^{-x}\left[\cos(x+y)(1)\right] = e^{-x}\cos(x+y) = \frac{\cos(x+y)}{e^x}$$

16) $f(x,y) = e^{xy} \ln y$

$$\frac{\partial f}{\partial x} = \ln y \left[e^{xy}(y) \right] = y e^{xy} \ln y$$

$$\frac{\partial f}{\partial y} = (e^{xy}) \left[\frac{1}{y}(1) \right] + (\ln y) \left[e^{xy}(x) \right] = \frac{e^{xy}}{y} + x e^{xy} \ln y$$

---

18) $f(x,y) = \cos^2(3x - y^2)$

$$\frac{\partial f}{\partial x} = 2\cos(3x-y^2)(-\sin(3x-y^2))(3) = -6\cos(3x-y^2)\sin(3x-y^2)$$

$$\frac{\partial f}{\partial y} = 2\cos(3x-y^2)(-\sin(3x-y^2))(-2y) = -4y\cos(3x-y^2)\sin(3x-y^2)$$

---

20) $f(x,y) = \log_y x = \dfrac{\ln x}{\ln y} = \ln x (\ln y)^{-1}$

$$\frac{\partial f}{\partial x} = \frac{1}{\ln y}\left[\frac{1}{x}(1) \right] = \frac{1}{x \ln y}$$

$$\frac{\partial f}{\partial y} = \ln x \left[-1 (\ln y)^{-2} \left(\frac{1}{y}(1) \right) \right] = \frac{-\ln x}{y (\ln y)^2}$$

24) $f(x,y,z) = xy + yz + xz$

$$f_x = \frac{\partial f}{\partial x} = y[1] + 0 + z[1] = y + z$$

$$f_y = \frac{\partial f}{\partial y} = x[1] + z[1] + 0 = x + z$$

$$f_z = \frac{\partial f}{\partial z} = 0 + y[1] + x[1] = y + x$$

---

26) $f(x,y,z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$

$$f_x = \frac{\partial f}{\partial x} = \frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x+0+0) = -x(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{-x}{(\sqrt{x^2+y^2+z^2})^3}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(0+2y+0) = -y(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{-y}{(\sqrt{x^2+y^2+z^2})^3}$$

$$f_z = \frac{\partial f}{\partial z} = \frac{-1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(0+0+2z) = -z(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{-z}{(\sqrt{x^2+y^2+z^2})^3}$$

---

28) $f(x,y,z)\ \sec^{-1}(x+yz)$

$$f_x = \frac{\partial f}{\partial x} = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}(1+0) = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}(0+z[1]) = \frac{z}{|x+yz|\sqrt{(x+yz)^2-1}}$$

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}}(0+y[1]) = \frac{y}{|x+yz|\sqrt{(x+yz)^2-1}}$$

30) $f(x,y,z) = yz\ln(xy)$

$$f_x = \frac{\partial f}{\partial x} = yz\left[\frac{1}{xy}(y)\right] = \frac{yz}{x}$$

$$f_y = \frac{\partial f}{\partial y} = (yz)\left[\frac{1}{xy}(x)\right] + (\ln(xy))\left[z\right] = z + z\ln(xy)$$

$$f_z = \frac{\partial f}{\partial z} = y\ln(xy)\left[1\right] = y\ln(xy)$$

---

32) $f(x,y,z) = e^{-xyz}$

$$f_x = \frac{\partial f}{\partial x} = e^{-xyz}(-yz) = -yz\,e^{-xyz} = \frac{-yz}{e^{xyz}}$$

$$f_y = \frac{\partial f}{\partial y} = e^{-xyz}(-xz) = -xz\,e^{-xyz} = \frac{-xz}{e^{xyz}}$$

$$f_z = \frac{\partial f}{\partial z} = e^{-xyz}(-xy) = -xy\,e^{-xyz} = \frac{-xy}{e^{xyz}}$$

---

34) $f(x,y,z) = \sinh(xy - z^2)$

$$f_x = \frac{\partial f}{\partial x} = \cosh(xy - z^2)(y-0) = y\cosh(xy - z^2)$$

$$f_y = \frac{\partial f}{\partial y} = \cosh(xy - z^2)(x-0) = x\cosh(xy - z^2)$$

$$f_z = \frac{\partial f}{\partial z} = \cosh(xy - z^2)(0-2z) = -2z\cosh(xy - z^2)$$

36) $g(u,v) = v^2 e^{\frac{2u}{v}}$

$$\frac{\partial g}{\partial u} = v^2\left[e^{\frac{2u}{v}}\left(\frac{2}{v}\right)\right] = 2v\, e^{\frac{2u}{v}}$$

$$\frac{\partial g}{\partial v} = (v^2)\left[e^{\frac{2u}{v}}\left(2u\left[-1v^{-2}\right]\right)\right] + \left(e^{\frac{2u}{v}}\right)\left[2v\right] = (v^2)\left[\frac{-2u}{v^2}e^{\frac{2u}{v}}\right] + \left(e^{\frac{2u}{v}}\right)\left[2v\right]$$

$$= -2u\, e^{\frac{2u}{v}} + 2v\, e^{\frac{2u}{v}} = 2v\, e^{\frac{2u}{v}} - 2u\, e^{\frac{2u}{v}}$$

---

38) $g(r,\theta,z) = r(1-\cos\theta) - z = r - r\cos\theta - z$

$$\frac{\partial g}{\partial r} = (1-\cos\theta)[1] - 0 = 1-\cos\theta$$

$$\frac{\partial g}{\partial \theta} = 0 - r[-\sin\theta(1)] - 0 = r\sin\theta$$

$$\frac{\partial g}{\partial z} = 0 - [1] = -1$$

---

42) $f(x,y) = \sin xy$

$$\frac{\partial f}{\partial x} = \cos(xy)(y) = y\cos(xy) \qquad \frac{\partial f}{\partial y} = \cos(xy)(x) = x\cos(xy)$$

$$\frac{\partial^2 f}{\partial x^2} = y\left[-\sin(xy)(y)\right] = -y^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = x\left[-\sin(xy)(x)\right] = -x^2\sin(xy)$$

$$\frac{\partial^2 f}{\partial y\,\partial x} = (y)\left[-\sin(xy)(x)\right] + (\cos(xy))[1] = \cos(xy) - xy\sin(xy)$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = (x)\left[-\sin(xy)(y)\right] + (\cos(xy))[1] = \cos(xy) - xy\sin(x,y)$$

44) $h(x,y) = xe^y + y + 1$

$\frac{\partial h}{\partial x} = e^y[1] + 0 + 0 = e^y$ $\qquad$ $\frac{\partial h}{\partial y} = x[e^y(1)] + [1] + 0 = xe^y + 1$

$\frac{\partial^2 h}{\partial x^2} = 0$ $\qquad$ $\frac{\partial^2 h}{\partial y^2} = x[e^y(1)] + 0 = xe^y$

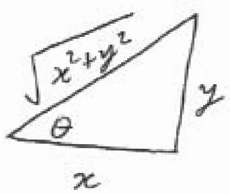
$\frac{\partial^2 h}{\partial y \partial x} = e^y(1) = e^y$ $\qquad$ $\frac{\partial^2 h}{\partial x \partial y} = e^y[1] + 0 = e^y$

---

46) $s(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\tan\theta = \frac{y}{x}$



$\sec^2\theta \frac{d\theta}{dx} = y[-1x^{-2}]$

$\frac{d\theta}{dx} = \frac{-y}{x^2\sec^2\theta} = \frac{-y}{x^2\left(\frac{\sqrt{x^2+y^2}}{x}\right)^2} = \frac{-y}{x^2+y^2}$

$\sec^2\theta \frac{d\theta}{dy} = \frac{1}{x}$

$\frac{d\theta}{dy} = \frac{1}{x\sec^2\theta} = \frac{1}{x\left(\frac{\sqrt{x^2+y^2}}{x}\right)^2} = \frac{1}{\frac{x^2+y^2}{x}} = \frac{x}{x^2+y^2}$

$\frac{\partial s}{\partial x} = \frac{-y}{x^2+y^2} = -y(x^2+y^2)^{-1}$ $\qquad$ $\frac{\partial s}{\partial y} = \frac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$

$\frac{\partial^2 s}{\partial x^2} = -y\left[-1(x^2+y^2)^{-2}(2x)\right] = \frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial^2 s}{\partial y^2} = x\left[-1(x^2+y^2)^{-2}(2y)\right] = \frac{-2xy}{(x^2+y^2)^2}$

$\frac{\partial^2 s}{\partial y \partial x} = \frac{(x^2+y^2)[-1] - (-y)[0+2y]}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial^2 s}{\partial x \partial y} = \frac{(x^2+y^2)[1] - (x)[2x+0]}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

48) $w = y e^{x^2-y}$

$$\frac{\partial w}{\partial x} = y\left[e^{x^2-y}(2x-0)\right] = 2xy\, e^{x^2-y}$$

$$\frac{\partial w}{\partial y} = (y)\left[e^{x^2-y}(0-1)\right] + \left(e^{x^2-y}\right)[1] = e^{x^2-y} - y\, e^{x^2-y} = (1-y)e^{x^2-y}$$

$$\frac{\partial^2 w}{\partial x^2} = (2xy)\left[e^{x^2-y}(2x-0)\right] + \left(e^{x^2-y}\right)[2y] = 4x^2y\, e^{x^2-y} + 2y\, e^{x^2-y}$$
$$= 2y\left(2x^2+1\right)e^{x^2-y}$$

$$\frac{\partial^2 w}{\partial y^2} = (1-y)\left[e^{x^2-y}(0-1)\right] + \left(e^{x^2-y}\right)[0-1] = -e^{x^2-y} + y e^{x^2-y} - e^{x^2-y}$$
$$= y e^{x^2-y} - 2e^{x^2-y} = (y-2)\,e^{x^2-y}$$

$$\frac{\partial^2 w}{\partial y\, \partial x} = (2xy)\left[e^{x^2-y}(0-1)\right] + \left(e^{x^2-y}\right)[2x] = 2x e^{x^2-y} - 2xy\, e^{x^2-y}$$
$$= 2x(1-y)\,e^{x^2-y}$$

$$\frac{\partial^2 w}{\partial x\, \partial y} = (1-y)\left[e^{x^2-y}(2x-0)\right] = 2x(1-y)\,e^{x^2-y}$$

---

50) $w = \dfrac{x-y}{x^2+y}$

$$\frac{\partial w}{\partial x} = \frac{(x^2+y)[1-0] - (x-y)[2x+0]}{(x^2+y)^2} = \frac{x^2+y-2x^2+2xy}{(x^2+y)^2} = \frac{-x^2+2xy+y}{(x^2+y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{(x^2+y)[0-1] - (x-y)[0+1]}{(x^2+y)^2} = \frac{-x^2-y-x+y}{(x^2+y)^2} = \frac{-x^2-x}{(x^2+y)^2} = (-x^2-x)(x^2+y)^{-2}$$

50) continued

$$\frac{\partial^2 w}{\partial x^2} = \frac{((x^2+y)^2)[-2x+2y] - (-x^2+2xy+y)[2(x^2+y)'(2x)]}{((x^2+y)^2)^2}$$

$$= \frac{2(x^2+y)\{(x^2+y)[-x+y] - (-x^2+2xy+y)[2x]\}}{(x^2+y)^4}$$

$$= \frac{2\{-x^3 - xy + x^2y + y^2 + 2x^3 - 4x^2y - 2xy\}}{(x^2+y)^3}$$

$$= \frac{2\{x^3 - 3x^2y - 3xy + y^2\}}{(x^2+y)^3}$$

$$\frac{\partial^2 w}{\partial y^2} = (-x^2-x)\left[-2(x^2+y)^{-3}(0+1)\right] = \frac{2(x^2+x)}{(x^2+y)^3} = \frac{2x^2+2x}{(x^2+y)^3}$$

$$\frac{\partial^2 w}{\partial y\,\partial x} = \frac{((x^2+y)^2)[0+2x+1] - (-x^2+2xy+y)[2(x^2+y)'(0+1)]}{((x^2+y)^2)^2}$$

$$= \frac{(x^2+y)\{(x^2+y)[2x+1] - (-x^2+2xy+y)[2]\}}{(x^2+y)^4}$$

$$= \frac{2x^3+2xy+x^2+y+2x^2-4xy-2y}{(x^2+y)^3} = \frac{2x^3+3x^2-2xy-y}{(x^2+y)^3}$$

$$\frac{\partial^2 w}{\partial x\,\partial y} = \frac{((x^2+y)^2)[-2x-1] - (-x^2-x)[2(x^2+y)'(2x)]}{((x^2+y)^2)^2}$$

$$= \frac{(x^2+y)\{(x^2+y)[-2x-1] - (-x^2-x)[4x]\}}{(x^2+y)^4}$$

$$= \frac{-2x^3-2xy-x^2-y+4x^3+4x^2}{(x^2+y)^3} = \frac{2x^3+3x^2-2xy-y}{(x^2+y)^3}$$

52) $g(x,y) = \cos(x^2) - \sin(3y)$

$$\frac{\partial g}{\partial x} = \left[-\sin(x^2)(2x)\right] - 0 = -2x\sin(x^2)$$

$$\frac{\partial g}{\partial y} = 0 - \left[\cos(3y)(3)\right] = -3\cos(3y)$$

$$\frac{\partial^2 g}{\partial x^2} = (-2x)\left[\cos(x^2)(2x)\right] + (\sin(x^2))\left[-2\right] = -4x^2\cos(x^2) - 2\sin(x^2)$$

$$\frac{\partial^2 g}{\partial y^2} = -3\left[-\sin(3y)(3)\right] = 9\sin(3y)$$

$$\frac{\partial^2 g}{\partial y\,\partial x} = 0 \qquad \frac{\partial^2 g}{\partial x\,\partial y} = 0$$

---

54) $z = xe^{\frac{x}{y^2}} = xe^{xy^{-2}}$

$$\frac{\partial z}{\partial x} = (x)\left[e^{\frac{x}{y^2}}\left(\frac{1}{y^2}\right)\right] + \left(e^{\frac{x}{y^2}}\right)\left[1\right] = \frac{x}{y^2}e^{\frac{x}{y^2}} + e^{\frac{x}{y^2}} = \left(\frac{x}{y^2}+1\right)e^{\frac{x}{y^2}}$$

$$\frac{\partial z}{\partial y} = x\left[e^{\frac{x}{y^2}}\left(x(-2y^{-3})\right)\right] = \frac{-2x^2}{y^3}e^{\frac{x}{y^2}} = \frac{-2x^2e^{\frac{x}{y^2}}}{y^3}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{x}{y^2}+1\right)\left[e^{\frac{x}{y^2}}\left(\frac{1}{y^2}\right)\right] + \left(e^{\frac{x}{y^2}}\right)\left[\frac{1}{y^2}\right] = \frac{1}{y^2}\left\{\left(\frac{x}{y^2}+1\right)+1\right\}e^{\frac{x}{y^2}}$$

$$= \frac{1}{y^2}\left\{\frac{x}{y^2}+2\right\}e^{\frac{x}{y^2}}$$

54) continued

$$\frac{\partial^2 z}{\partial y^2} = \frac{(y^3)\left[-2x^2 e^{\frac{x}{y^2}}(-2y^{-3})\right] - \left(-2x^2 e^{\frac{x}{y^2}}\right)\left[3y^2\right]}{(y^3)^2}$$

$$= \frac{(y^3)\left[\frac{4x^2}{y^3} e^{\frac{x}{y^2}}\right] - \left(-2x^2 e^{\frac{x}{y^2}}\right)\left[3y^2\right]}{y^6} = \frac{4x^2 e^{\frac{x}{y^2}} + 6x^2y^2 e^{\frac{x}{y^2}}}{y^6}$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = \left(\frac{x}{y^2}+1\right)\left[e^{\frac{x}{y^2}}\left(x(-2y^{-3})\right)\right] + \left(e^{\frac{x}{y^2}}\right)\left[x(-2y^{-3})\right]$$

$$= \left(\frac{x}{y^2}+1\right)\left[\frac{-2x}{y^3} e^{\frac{x}{y^2}}\right] + \left(e^{\frac{x}{y^2}}\right)\left[\frac{-2x}{y^3}\right]$$

$$= \frac{-2x}{y^3}\left\{\left(\frac{x}{y^2}+1\right)+1\right\} e^{\frac{x}{y^2}} = \frac{-2x}{y^3}\left\{\frac{x}{y^2}+2\right\} e^{\frac{x}{y^2}}$$

$$= \frac{-2x^2}{y^5} e^{\frac{x}{y^2}} - \frac{4x}{y^3} e^{\frac{x}{y^2}}$$

$$\frac{\partial^2 z}{\partial x\, \partial y} = \left(\frac{-2x^2}{y^3}\right)\left[e^{\frac{x}{y^2}}\left(\frac{1}{y^2}(1)\right)\right] + \left(e^{\frac{x}{y^2}}\right)\left[\frac{-2}{y^3}(2x)\right]$$

$$= \frac{-2x^2}{y^5} e^{\frac{x}{y^2}} - \frac{4x}{y^3} e^{\frac{x}{y^2}}$$

$$= \frac{-2x}{y^3}\left\{\frac{x}{y^2}+2\right\} e^{\frac{x}{y^2}}$$

56) $w = e^{x} + x\ln y + y\ln x$

$$\frac{\partial w}{\partial x} = [e^{x}(1)] + \ln y\,[1] + y\left[\frac{1}{x}(1)\right] = e^{x} + \ln y + \frac{y}{x}$$

$$\frac{\partial w}{\partial y} = 0 + x\left[\frac{1}{y}(1)\right] + \ln x\,[1] = \frac{x}{y} + \ln x$$

$$\frac{\partial^2 w}{\partial y\,\partial x} = 0 + \left[\frac{1}{y}(1)\right] + \frac{1}{x}[1] = \frac{1}{y} + \frac{1}{x}$$

$$\frac{\partial^2 w}{\partial x\,\partial y} = \frac{1}{y}[1] + \left[\frac{1}{x}(1)\right] = \frac{1}{y} + \frac{1}{x}$$

---

58) $w = x\sin y + y\sin x + xy$

$$\frac{\partial w}{\partial x} = \sin y\,[1] + y[\cos x(1)] + y\,[1] = \sin y + y\cos x + y$$

$$\frac{\partial w}{\partial y} = x[\cos y(1)] + \sin x\,[1] + x\,[1] = x\cos y + \sin x + x$$

$$\frac{\partial^2 w}{\partial y\,\partial x} = [\cos y(1)] + \cos x\,[1] + [1] = \cos y + \cos x + 1$$

$$\frac{\partial^2 w}{\partial x\,\partial y} = \cos y\,[1] + [\cos x(1)] + [1] = \cos y + \cos x + 1$$

60) $w = \dfrac{3x - y}{x + y}$

$$\frac{\partial w}{\partial x} = \frac{(x+y)[3] - (3x-y)[1]}{(x+y)^2} = \frac{3x + 3y - 3x + y}{(x+y)^2} = \frac{4y}{(x+y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{(x+y)[-1] - (3x-y)[1]}{(x+y)^2} = \frac{-x - y - 3x + y}{(x+y)^2} = \frac{-4x}{(x+y)^2}$$

$$\frac{\partial^2 w}{\partial y \partial x} = \frac{((x+y)^2)[4] - (4y)[2(x+y)'(1)]}{((x+y)^2)^2} = \frac{4(x+y)\{(x+y)[1] - (y)[2]\}}{(x+y)^4}$$

$$= \frac{4\{x + y - 2y\}}{(x+y)^3} = \frac{4\{x - y\}}{(x+y)^3}$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{((x+y)^2)[-4] - (-4x)[2(x+y)'(1)]}{((x+y)^2)^2} = \frac{4(x+y)\{(x+y)[-1] - (-x)[2]\}}{(x+y)^4}$$

$$= \frac{4\{-x - y + 2x\}}{(x+y)^3} = \frac{4\{x - y\}}{(x+y)^3}$$

---

84) $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z = 2z^3 - 3x^2z - 3y^2z$

$$\frac{\partial f}{\partial x} = 0 - 3z[2x] - 0 = -6xz \qquad \frac{\partial f}{\partial y} = 0 - 0 - 3z[2y] = -6yz$$

$$\frac{\partial f}{\partial z} = 2[3z^2] - 3(x^2 + y^2)[1] = 6z^2 - 3(x^2 + y^2)$$

$$\frac{\partial^2 f}{\partial x^2} = -6z[1] = -6z \qquad \frac{\partial^2 f}{\partial y^2} = -6z[1] = -6z \qquad \frac{\partial^2 f}{\partial z^2}\ 6[2z] - 0 = 12z$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = (-6z) + (-6z) + (12z) = 0$$

86) $f(x,y) = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+y^2)$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left[\frac{1}{x^2+y^2}(2x)\right] = \frac{x}{x^2+y^2} \qquad \frac{\partial f}{\partial y} = \frac{1}{2}\left[\frac{1}{x^2+y^2}(2y)\right] = \frac{y}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2)[1]-(x)[2x]}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)[1]-(y)[2y]}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) + \left(\frac{x^2-y^2}{(x^2+y^2)^2}\right) = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$$

---

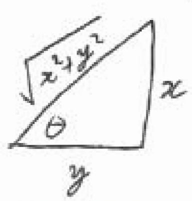
88) $f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$

$$\sec^2\theta\,\frac{d\theta}{dx} = \frac{1}{y}$$

$$\frac{d\theta}{dx} = \frac{1}{y\sec^2\theta} = \frac{1}{y\left(\frac{\sqrt{x^2+y^2}}{y}\right)^2} = \frac{y}{x^2+y^2}$$

$\theta = \tan^{-1}\left(\frac{x}{y}\right)$

$\tan\theta = \frac{x}{y}$



$$\sec^2\theta\,\frac{d\theta}{dy} = x\left[-1y^{-2}\right] = \frac{-x}{y^2}$$

$$\frac{d\theta}{dy} = \frac{-x}{y^2\sec^2\theta} = \frac{-x}{y^2\left(\frac{\sqrt{x^2+y^2}}{y}\right)^2} = \frac{-x}{x^2+y^2}$$

$$\frac{\partial f}{\partial x} = \frac{y}{x^2+y^2} = y(x^2+y^2)^{-1} \qquad \frac{\partial f}{\partial y} = \frac{-x}{x^2+y^2} = -x(x^2+y^2)^{-1}$$

$$\frac{\partial^2 f}{\partial x^2} = y\left[-1(x^2+y^2)^{-2}(2x)\right] = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = -x\left[-1(x^2+y^2)^{-2}(2y)\right] = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{-2xy}{(x^2+y^2)^2}\right) + \left(\frac{2xy}{(x^2+y^2)^2}\right) = 0$$

90) $f(x,y,z) = e^{3x+4y}\cos(5z)$

$$\frac{\partial f}{\partial x} = \cos(5z)\left[e^{3x+4y}(3)\right] = 3e^{3x+4y}\cos(5z)$$

$$\frac{\partial f}{\partial y} = \cos(5z)\left[e^{3x+4y}(4)\right] = 4e^{3x+4y}\cos(5z)$$

$$\frac{\partial f}{\partial z} = e^{3x+4y}\left[-\sin(5z)(5)\right] = -5e^{3x+4y}\sin(5z)$$

$$\frac{\partial^2 f}{\partial x^2} = 3\cos(5z)\left[e^{3x+4y}(3)\right] = 9e^{3x+4y}\cos(5z)$$

$$\frac{\partial^2 f}{\partial y^2} = 4\cos(5z)\left[e^{3x+4y}(4)\right] = 16e^{3x+4y}\cos(5z)$$

$$\frac{\partial^2 f}{\partial z^2} = -5e^{3x+4y}\left[\cos(5z)(5)\right] = -25e^{3x+4y}\cos(5z)$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \left(9e^{3x+4y}\cos(5z)\right) + \left(16e^{3x+4y}\cos(5z)\right) + \left(-25e^{3x+4y}\cos(5z)\right)$$

$$= 0$$

92) $w = \cos(2x + 2ct)$

$$\frac{\partial w}{\partial x} = -\sin(2x+2ct)(2) = -2\sin(2x+2ct)$$

$$\frac{\partial w}{\partial t} = -\sin(2x+2ct)(2c) = -2c\sin(2x+2ct)$$

$$\frac{\partial^2 w}{\partial x^2} = -2\left[\cos(2x+2ct)(2)\right] = -4\cos(2x+2ct)$$

$$\frac{\partial^2 w}{\partial t^2} = -2c\left[\cos(2x+2ct)(2c)\right] = -4c^2\cos(2x+2ct)$$

$$\frac{\partial^2 w}{\partial t^2} = -4c^2\cos(2x+2ct) = c^2\{-4\cos(2x+2ct)\} = c^2\frac{\partial^2 w}{\partial x^2}$$

---

94) $w = \ln(2x+2ct)$

$$\frac{\partial w}{\partial x} = \frac{1}{2x+2ct}(2) = \frac{2}{2x+2ct} = \frac{1}{x+ct} = (x+ct)^{-1} \qquad \frac{\partial w}{\partial t} = \frac{1}{2x+2ct}(2c) = \frac{2c}{2x+2ct} = \frac{c}{x+ct} = c(x+ct)^{-1}$$

$$\frac{\partial^2 w}{\partial x^2} = -1(x+ct)^{-2}(1) = \frac{-1}{(x+ct)^2}$$

$$\frac{\partial^2 w}{\partial t^2} = c\left[-1(x+ct)^{-2}(c)\right] = \frac{-c^2}{(x+ct)^2}$$

$$\frac{\partial^2 w}{\partial t^2} = \frac{-c^2}{(x+ct)^2} = c^2\left\{\frac{-1}{(x+ct)^2}\right\} = c^2\frac{\partial^2 w}{\partial x^2}$$