Limits for Functions of Two Variables

Definitions

We say that a function $f(x,y)$ approaches the **limit** L as (x,y) approaches (x_0,y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$$

if, for every number $\varepsilon>0$, there exists a corresponding number $\delta>0$ such that for all (x,y) in the domain of f,

$$\left|f(x,y)-L\right|<\varepsilon \quad \text{whenever} \quad 0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta .$$

Theorem 1 – Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y)=M .$$

1. *Sum Rule*: $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y)+g(x,y)\right)=L+M$

2. *Difference Rule*: $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y)-g(x,y)\right)=L-M$

3. *Constant Multiple Rule*: $\lim_{(x,y)\to(x_0,y_0)} kf(x,y)=kL$ (any number k)

4: *Product Rule*: $\lim_{(x,y)\to(x_0,y_0)} \left(f(x,y)\right)\left(g(x,y)\right)=(L)(M)$

5. *Quotient Rule*: $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)}=\dfrac{L}{M}$ $M\neq 0$

6. *Power Rule*: $\lim_{(x,y)\to(x_0,y_0)} \left[f(x,y)\right]^n=L^n$ n a positive integer

7. *Root Rule*: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)}=\sqrt[n]{L}=L^{1/n}$ n a positive integer,
and if n even, we assume that $L>0$

Continuity

Definition

A function $f(x, y)$ is **continuous at the point** (x_0, y_0) if

1. f is defined at (x_0, y_0),
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Having the same limit along all straight lines approaching (x_0, y_0) does not imply that a limit exists at (x_0, y_0).

Continuity of Compositions

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composition $h = g \circ f$ defined by $h(x, y) - g(f(x, y))$ is continuous at (x_0, y_0).

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2) $\lim_{(x,y)\to(0,4)} \frac{x}{\sqrt{y}} = \frac{(0)}{\sqrt{(4)}} = \frac{0}{2} = 0$

4) $\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x}+\frac{1}{y}\right)^2 = \left(\frac{1}{(2)}+\frac{1}{(-3)}\right)^2 = \left(\frac{1}{2}-\frac{1}{3}\right)^2 = \left(\frac{1}{6}\right)^2 = \frac{1}{36}$

6) $\lim_{(x,y)\to(0,0)} \cos\left(\frac{x^2+y^3}{x+y+1}\right) = \cos\left(\frac{(0)^2+(0)^3}{(0)+(0)+1}\right) = \cos(0) = 1$

8) $\lim_{(x,y)\to(1,1)} \ln|1+x^2y^2| = \ln|1+(1)^2(1)^2| = \ln|1+1| = \ln|2| = \ln 2$

10) $\lim_{(x,y)\to(\frac{1}{27},\pi^3)} \cos\sqrt[3]{xy} = \cos\sqrt[3]{(\tfrac{1}{27})(\pi^3)} = \cos\sqrt[3]{\frac{\pi^3}{3^3}} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

12) $\lim_{(x,y)\to(\frac{\pi}{2},0)} \frac{\cos y+1}{y-\sin x} = \frac{\cos(0)+1}{(0)-\sin(\frac{\pi}{2})} = \frac{1+1}{0-(1)} = \frac{2}{-1} = -2$

14) $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-y^2}{x-y} = \lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y)\to(1,1)} (x+y) = (1)+(1) = 2$

16) $\lim_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\ x\neq x^2}} \frac{y+4}{x^2y-xy+4x^2-4x} = \lim_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\ x\neq x^2}} \frac{y+4}{x\{y(x-1)+4(x-1)\}} = \lim_{\substack{(x,y)\to(2,-4)\\ y\neq -4,\ x\neq x^2}} \frac{y+4}{x\{y+4\}(x-1)}$

$= \lim_{\substack{(x,y)\to(2,-4)\\ x\neq x^2}} \frac{1}{x(x-1)} = \frac{1}{(2)((2)-1)} = \frac{1}{2(1)} = \frac{1}{2}$

↑ error in text

18) $$\lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{x+y-4}{\sqrt{x+y}-2} = \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{(\sqrt{x+y})^2-(2)^2}{\sqrt{x+y}-2} = \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \frac{(\sqrt{x+y}+2)(\sqrt{x+y}-2)}{\sqrt{x+y}-2}$$

$$= \lim_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} (\sqrt{x+y}+2) = \sqrt{(2)+(2)}+2 = \sqrt{4}+2 = 2+2 = 4$$

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20) $$\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1} = \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-(y+1)} = \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{(\sqrt{x}+\sqrt{y+1})(\sqrt{x}-\sqrt{y+1})}$$

$$= \lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{1}{\sqrt{x}+\sqrt{y+1}} = \frac{1}{\sqrt{(4)}+\sqrt{(3)+1}} = \frac{1}{\sqrt{4}+\sqrt{4}} = \frac{1}{2+2} = \frac{1}{4}$$

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22) $$\lim_{(x,y)\to(0,0)} \frac{1-\cos(xy)}{xy} = \lim_{u\to 0} \frac{1-\cos u}{u} \underline{\underline{L}} \lim_{u\to 0} \frac{\sin u}{1} = \frac{\sin(0)}{1} = \frac{0}{1} = 0$$

let $xy = u$

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24) $$\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \lim_{(x,y)\to(2,2)} \frac{x-y}{(x^2+y^2)(x^2-y^2)} = \lim_{(x,y)\to(2,2)} \frac{x-y}{(x^2+y^2)(x+y)(x-y)}$$

$$= \lim_{(x,y)\to(2,2)} \frac{1}{(x^2+y^2)(x+y)} = \frac{1}{((2)^2+(2)^2)((2)+(2))} = \frac{1}{(4+4)(4)} = \frac{1}{8(4)} = \frac{1}{32}$$

32-a) $f(x,y) = \frac{x+y}{x-y}$ all (x,y) such that $x \neq y$

32-b) $f(x,y) = \frac{y}{x^2-1}$ all (x,y)

34-a) $g(x,y) = \frac{x^2+y^2}{x^2-3x+2}$ all (x,y) such that $x^2-3x+2 \neq 0$

$(x-2)(x-1) \neq 0$

$x-2 \neq 0 \mid x-1 \neq 0$

$x \neq 2 \mid x \neq 1$

34-b) $g(x,y) = \frac{1}{x^2-y}$ all (x,y) such that $y \neq x^2$

36-a) $f(x,y,z) = \ln xyz$ all (x,y,z) such that $xyz > 0$

36-b) $f(x,y,z) = e^{x+y}\cos z$ all (x,y,z)

38-a) $h(x,y,z) = \frac{1}{|y|+|z|}$ all (x,y,z) except $(x,0,0)$

38-b) $h(x,y,z) = \frac{1}{|xy|+|z|}$ all (x,y,z) except $(0,y,0)$ or $(x,0,0)$

40-a) $h(x,y,z) = \sqrt{4-x^2-y^2-z^2}$ all (x,y,z) such that $x^2+y^2+z^2 \le 4$

40-b) $h(x,y,z) = \frac{1}{4-\sqrt{x^2+y^2+z^2-9}}$ all (x,y,z) such that $x^2+y^2+z^2 \ge 9$

except when $x^2+y^2+z^2 = 25$

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42) $f(x,y) = \frac{x^4}{x^4+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=0}} \frac{x^4}{x^4+y^2} = \lim_{x\to 0} \frac{x^4}{x^4+0^2} = \lim_{x\to 0}\frac{x^4}{x^4} = \lim_{x\to 0} 1 = 1$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x^2}} \frac{x^4}{x^4+y^2} = \lim_{x\to 0} \frac{x^4}{x^4+(x^2)^2} = \lim_{x\to 0}\frac{x^4}{x^4+x^4} = \lim_{x\to 0}\frac{x^4}{2x^4} = \lim_{x\to 0}\frac{1}{2} = \frac{1}{2}$$

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44) $f(x,y) = \frac{xy}{|xy|}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=kx\\ k\neq 0}} \frac{xy}{|xy|} = \lim_{x\to 0}\frac{x(kx)}{|x(kx)|} = \lim_{x\to 0}\frac{kx^2}{|kx^2|} = \lim_{x\to 0}\frac{kx^2}{|k|x^2} = \lim_{x\to 0}\frac{k}{|k|}$$

if $k>0$, the limit is 1

if $k<0$, the limit is -1

46) $g(x,y) = \dfrac{x^2 - y}{x - y}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y = kx\\ k\neq 1}} \frac{x^2-y}{x-y} = \lim_{x\to 0}\frac{x^2-(kx)}{x-(kx)} = \lim_{x\to 0}\frac{x(x-k)}{x(1-k)} = \lim_{x\to 0}\frac{x-k}{1-k} = \frac{(0)-k}{1-k} = \frac{-k}{1-k}$$

we will get different limits for different values of k, $k \neq 1$

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48) $h(x,y) = \dfrac{x^2 y}{x^4 + y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y = kx^2}} \frac{x^2 y}{x^4+y^2} = \lim_{x\to 0}\frac{x^2(kx^2)}{x^4+(kx^2)^2} = \lim_{x\to 0}\frac{kx^4}{x^4+k^2x^4} = \lim_{x\to 0}\frac{kx^4}{x^4(1+k^2)}$$

$$= \lim_{x\to 0}\frac{k}{1+k^2} = \frac{k}{1+k^2}$$

different limits for different values of k

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50) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$

$$\lim_{\substack{(x,y)\to(1,-1)\\ \text{along } y=-1}} \frac{xy+1}{x^2-y^2} = \lim_{x\to 1}\frac{x(-1)+1}{x^2-(-1)^2} = \lim_{x\to 1}\frac{-x+1}{x^2-1} = \lim_{x\to 1}\frac{-(x-1)}{(x+1)(x-1)}$$

$$= \lim_{x\to 1}\frac{-1}{x+1} = \frac{-1}{(1)+1} = \frac{-1}{2}$$

$$\lim_{\substack{(x,y)\to(1,-1)\\ \text{along } y=-x^2}} \frac{xy+1}{x^2-y^2} = \lim_{x\to 1}\frac{x(-x^2)+1}{x^2-(-x^2)^2} = \lim_{x\to 1}\frac{-x^3+1}{x^2-x^4} = \lim_{x\to 1}\frac{1-x^3}{x^2(1-x^2)}$$

$$= \lim_{x\to 1}\frac{(1-x)(1+x+x^2)}{x^2(1+x)(1-x)} = \lim_{x\to 1}\frac{1+x+x^2}{x^2(1+x)} = \frac{1+(1)+(1)^2}{(1)^2(1+(1))} = \frac{3}{1(2)} = \frac{3}{2}$$

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52) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{xe^y-1}{xe^y-1+y}$

$$\lim_{\substack{(x,y)\to(1,0)\\ \text{along } y=0}} \frac{xe^y-1}{xe^y-1+y} = \lim_{x\to 1}\frac{xe^{(0)}-1}{xe^{(0)}-1+(0)} = \lim_{x\to 1}\frac{x-1}{x-1} = \lim_{x\to 1} 1 = 1$$

52) continued

$$\lim_{\substack{(x,y)\to(1,0)\\ \text{along } x=1}} \frac{xe^y-1}{xe^y-1+y} = \lim_{y\to 0} \frac{(1)e^y-1}{(1)e^y-1+y} = \lim_{y\to 0}\frac{e^y-1}{e^y-1+y} \stackrel{L}{=} \lim_{y\to 0}\frac{e^y}{e^y+1}$$

$$= \frac{e^{(0)}}{e^{(0)}+1} = \frac{1}{1+1} = \frac{1}{2}$$

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54) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{\tan y - y\tan x}{y-x}$

$$\lim_{\substack{(x,y)\to(1,1)\\ \text{along } y=1}} \frac{\tan y - y\tan x}{y-x} = \lim_{x\to 1}\frac{\tan(1)-(1)\tan x}{(1)-x} = \lim_{x\to 1}\frac{\tan 1-\tan x}{1-x}$$

$$\stackrel{L}{=} \lim_{x\to 1}\frac{-\sec^2 x}{-1} = \lim_{x\to 1}\sec^2 x = \sec^2(1)$$

$$\lim_{\substack{(x,y)\to(1,1)\\ \text{along } x=1}} \frac{\tan y - y\tan x}{y-x} = \lim_{y\to 1}\frac{\tan y - y\tan(1)}{y-(1)} = \lim_{y\to 1}\frac{\tan y - y\tan 1}{y-1}$$

$$\stackrel{L}{=} \lim_{y\to 1}\frac{\sec^2 y - \tan 1}{1} = \lim_{y\to 1}\sec^2 y - \tan 1 = \sec^2(1) - \tan 1$$

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60) $2|xy| - \frac{x^2y^2}{6} < 4 - 4\cos\sqrt{|xy|} < 2|xy|$; $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|} = ?$

if $xy>0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2|xy| - \frac{x^2y^2}{6}}{|xy|} = \lim_{(x,y)\to(0,0)} \frac{2xy - \frac{x^2y^2}{6}}{xy} = \lim_{(x,y)\to(0,0)} \frac{xy\left(2-\frac{xy}{6}\right)}{xy}$

$$= \lim_{(x,y)\to(0,0)}\left(2-\frac{xy}{6}\right) = 2 - \frac{(0)(0)}{6} = 2-0 = 2$$

and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2|xy|}{|xy|} = \lim_{(x,y)\to(0,0)} 2 = 2$

60) continued

if $xy<0$, $\lim_{(x,y)\to(0,0)} \frac{2|xy|-\left(\frac{x^2y^2}{6}\right)}{|xy|} = \lim_{(x,y)\to(0,0)} \frac{-2xy-\frac{x^2y^2}{6}}{-xy} = \lim_{(x,y)\to(0,0)} \frac{-xy\left(2+\frac{xy}{6}\right)}{-xy}$

$$= \lim_{(x,y)\to(0,0)} \left(2+\frac{xy}{6}\right) = 2+\frac{(0)(0)}{6} = 2+0 = 2$$

and $\lim_{(x,y)\to(0,0)} \frac{2|xy|}{|xy|} = \lim_{(x,y)\to(0,0)} 2 = 2$

$$2|xy| - \frac{x^2y^2}{6} < 4-4\cos\sqrt{|xy|} < 2|xy|$$

$$\frac{2|xy|-\frac{x^2y^2}{6}}{|xy|} < \frac{4-4\cos\sqrt{|xy|}}{|xy|} < \frac{2|xy|}{|xy|}$$

$$2 = \lim_{(x,y)\to(0,0)} \frac{2|xy|-\frac{x^2y^2}{6}}{|xy|} < \lim_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|} < \lim_{(x,y)\to(0,0)} \frac{2|xy|}{|xy|} = 2$$

by the Sandwich Theorem, $\lim_{(x,y)\to(0,0)} \frac{4-4\cos\sqrt{|xy|}}{|xy|} = 2$

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62) $\left|\cos\left(\frac{1}{y}\right)\right| \le 1$; $\lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{y}\right) = ?$

$-1 \le \cos\left(\frac{1}{y}\right) \le 1$

for $x \ge 0$

$-x \le x\cos\left(\frac{1}{y}\right) \le x$

$0 = \lim_{x\to 0}(-x) \le \lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{y}\right) \le \lim_{x\to 0} x = 0$

for $x \le 0$

$-x \ge x\cos\left(\frac{1}{y}\right) \ge x$

$0 = \lim_{x\to 0}(-x) \ge \lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{y}\right) \ge \lim_{x\to 0} x = 0$

Therefore by the Sandwich Theorem,

$$\lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{y}\right) = 0$$

72) $f(x,y)=\dfrac{3x^2y}{x^2+y^2}$ let $x=r\cos\theta$, $y=r\sin\theta$

so $x^2+y^2=(r\cos\theta)^2+(r\sin\theta)^2=r^2\cos^2\theta+r^2\sin^2\theta$
$=r^2(\cos^2\theta+\sin^2\theta)=r^2(1)=r^2$

$$\lim_{(x,y)\to(0,0)}\frac{3x^2y}{x^2+y^2}=\lim_{r\to 0}\frac{3(r\cos\theta)^2(r\sin\theta)}{r^2}=\lim_{r\to 0}\frac{3r^3\cos^2\theta\sin\theta}{r^2}$$
$$=\lim_{r\to 0}3r\cos^2\theta\sin\theta=3(0)\cos^2\theta\sin\theta=0$$

so define $f(0,0)=0$

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74) $f(x,y)=\dfrac{y}{x^2+1}$, $\varepsilon=0.05$

let $\delta=0.05$. Then $|x|<\delta$ and $|y|<\delta$

$$|f(x,y)-f(0,0)|=\left|\frac{y}{x^2+1}-0\right|=\left|\frac{y}{x^2+1}\right|\le|y|<0.05=\varepsilon$$

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76) $f(x,y)=\dfrac{x+y}{2+\cos x}$, $\varepsilon=0.02$

let $\delta=0.01$. Since $-1\le\cos x\le 1$

$$1\le 2+\cos x\le 3$$

$$\frac{1}{3}\le\frac{1}{2+\cos x}\le\frac{1}{1}=1$$

$$\frac{|x+y|}{3}\le\left|\frac{x+y}{2+\cos y}\right|\le|x+y|\le|x|+|y|.$$

Then $|x|<\delta$ and $|y|<\delta$

$$|f(x,y)-f(0,0)|=\left|\frac{x+y}{2+\cos x}-0\right|=\left|\frac{x+y}{2+\cos x}\right|\le|x|+|y|<0.01+0.01=0.02=\varepsilon$$

78) $f(x,y) = \frac{x^3+y^4}{x^2+y^2}$ and $f(0,0)=0$, $\varepsilon = 0.02$

let $\delta = 0.01$. If $|y| \le 1$, then $y^2 \le |y| = \sqrt{y^2} \le \sqrt{x^2+y^2}$,

so $|x| = \sqrt{x^2} \le \sqrt{x^2+y^2}$

$$|x| + y^2 \le \sqrt{x^2+y^2} + \sqrt{x^2+y^2} = 2\sqrt{x^2+y^2}$$

$$|x| + y^2 \le 2\sqrt{x^2+y^2}$$

Since $x^2 \le x^2+y^2$ and $y^2 \le x^2+y^2$

$$\frac{x^2}{x^2+y^2} \le \frac{x^2}{x^2} = 1 \qquad \frac{y^2}{x^2+y^2} \le \frac{y^2}{y^2} = 1$$

$$\frac{x}{x^2+y^2} \le 1 \qquad \frac{y^2}{x^2+y^2} \le 1$$

Then $\frac{|x^3+y^4|}{x^2+y^2} \le \frac{x^2}{x^2+y^2}|x| + \frac{y^2}{x^2+y^2}y^2 \le |x| + y^2 < 2\delta$

$$|f(x,y) - f(0,0)| = \left|\frac{x^3+y^4}{x^2+y^2} - 0\right| < 2(0.01) = 0.02 = \varepsilon$$