Definitions

Suppose D is a set of *n*-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of *w*-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

Functions of Two Variables

Definitions

A point (x_0, y_0) in a region (set) R in the *xy*-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (see figure in text, Figure 14.2 in 14th edition). A point (x_0, y_0) is a **boundary point** of R if for every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (see figure in text, Figure 14.3 in 14th edition).

Definitions

A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Graphs, Level Curves, and Contours of Functions of Two Variables

Definitions

The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

Functions of Three Variables

Definitions

The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

Definitions

A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if it is the center of a solid ball that lies entirely in R (see figure in text, Figure 14.9a in 14th edition). A point (x_0, y_0, z_0) is a **boundary point** of R if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside of R as well as points that lie inside (see figure in text, Figure 14.9b in 14th edition). The **interior** of R is the set of interior points of R. The **boundary** of R is the set of boundary points of R.

A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.

2) $f(x,y) = \sin(xy)$

a) $f\left(2,\frac{\pi}{6}\right) = \sin\left((2)\left(\frac{\pi}{6}\right)\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

b) $f\left(-3,\frac{\pi}{12}\right) = \sin\left((-3)\left(\frac{\pi}{12}\right)\right) = \sin\left(\frac{-\pi}{4}\right) = \frac{-1}{\sqrt{2}}$

c) $f\left(\pi,\frac{1}{4}\right) = \sin\left((\pi)\left(\frac{1}{4}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

d) $f\left(\frac{-\pi}{2},-7\right) = \sin\left(\left(\frac{-\pi}{2}\right)(-7)\right) = \sin\left(\frac{7\pi}{2}\right) = \sin\left(\frac{4\pi}{2}+\frac{3\pi}{2}\right) = \sin\left(2\pi+\frac{3\pi}{2}\right)$

$= \sin\left(\frac{3\pi}{2}\right) = -1$

---

4) $f(x,y,z) = \sqrt{49-x^2-y^2-z^2}$

a) $f(0,0,0) = \sqrt{49-(0)^2-(0)^2-(0)^2} = \sqrt{49} = 7$

b) $f(2,-3,6) = \sqrt{49-(2)^2-(-3)^2-(6)^2} = \sqrt{49-4-9-36} = \sqrt{0} = 0$

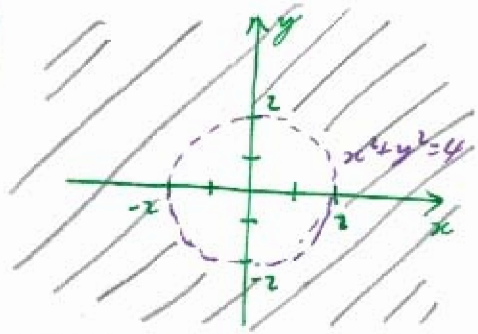
c) $f(-1,2,3) = \sqrt{49-(-1)^2-(2)^2-(3)^2} = \sqrt{49-1-4-9} = \sqrt{35}$

d) $f\left(\frac{4}{\sqrt{2}},\frac{5}{\sqrt{2}},\frac{6}{\sqrt{2}}\right) = \sqrt{49-\left(\frac{4}{\sqrt{2}}\right)^2-\left(\frac{5}{\sqrt{2}}\right)^2-\left(\frac{6}{\sqrt{2}}\right)^2} = \sqrt{49-\frac{16}{2}-\frac{25}{2}-\frac{36}{2}}$

$= \sqrt{49-8-\frac{25}{2}-18} = \sqrt{23-\frac{25}{2}} = \sqrt{\frac{46}{2}-\frac{25}{2}}$

$= \sqrt{\frac{21}{2}}$

6) $f(x,y) = \ln(x^2+y^2-4)$



$x^2+y^2-4>0$

$x^2+y^2>4$

Domain: all points (x,y) outside the circle $x^2+y^2=4$

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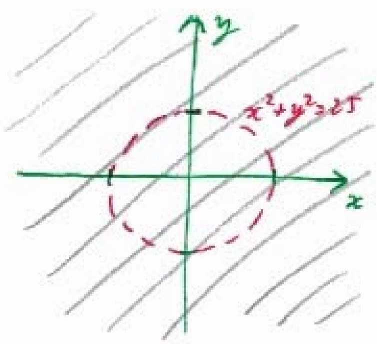
8) $f(x,y) = \dfrac{\sin(xy)}{x^2+y^2-25}$



numerator: all real numbers

denominator: not $x^2+y^2-25=0$

$x^2+y^2=25$

$x^2+y^2=(5)^2$

Domain: all points (x,y) not lying on the graph $x^2+y^2=25$

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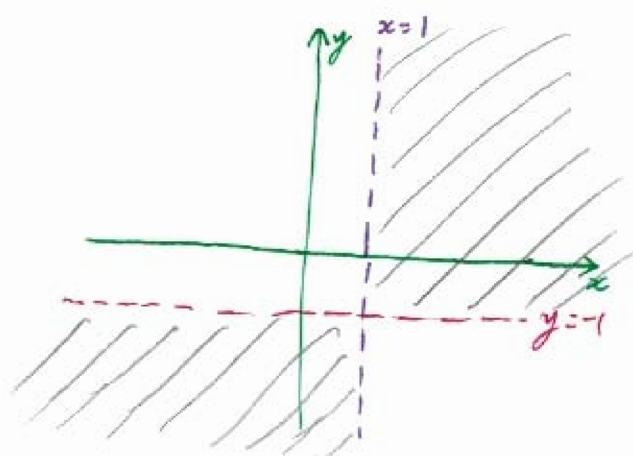
10) $f(x,y) = \ln(xy+x-y-1)$



$xy+x-y-1>0$ $\quad x-1=0$

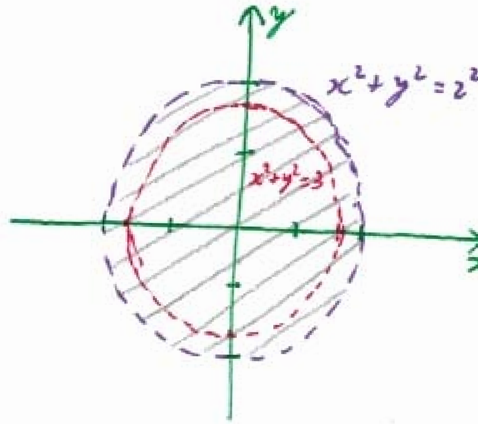
$x(y+1)-1(y+1)>0$ $\quad x=1$

$(x-1)(y+1)>0$ $\quad y+1=0$

$(x-1)(y+1)>0$ $\quad y=-1$

Domain: all points (x,y) satisfying $(x-1)(y+1)>0$

12) $f(x,y) = \dfrac{1}{\ln(4-x^2-y^2)}$



$4 - x^2 - y^2 > 0$

$4 > x^2+y^2$

$2^2 = x^2+y^2$

also,

$4 - x^2 - y^2 = 1$

$3 = x^2+y^2$

$(\sqrt{3})^2 = x^2+y^2$

Domain: all points (x,y) inside the circle $x^2+y^2=2^2$ such that $x^2+y^2 \neq 3$

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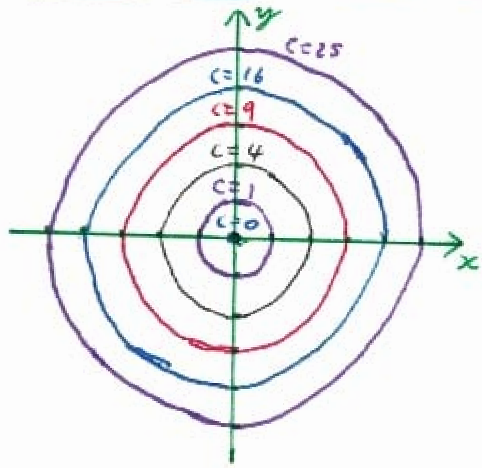
14) $f(x,y) = x^2+y^2$



$c = 0, 1, 4, 9, 16, 25$

$x^2+y^2=0$ "point at origin"

$x^2+y^2=1$

$x^2+y^2=4$

$x^2+y^2=9$

$x^2+y^2=16$

$x^2+y^2=25$

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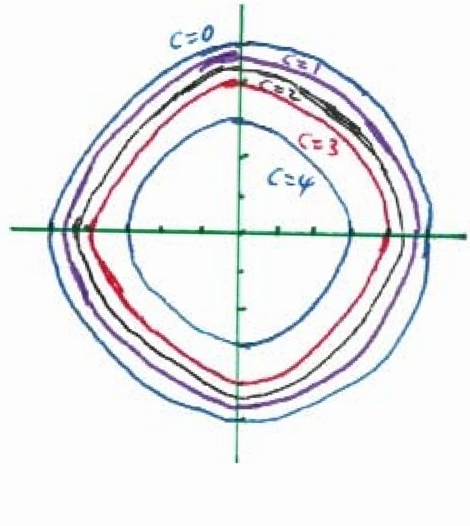
16) $f(x,y) = \sqrt{25-x^2-y^2}$

$c = 0, 1, 2, 3, 4$



$\sqrt{25-x^2-y^2} = 0$

$25-x^2-y^2=0$

$25 = x^2+y^2$

$\sqrt{25-x^2-y^2} = 1$

$25-x^2-y^2=1$

$24 = x^2+y^2$

$\sqrt{25-x^2-y^2} = 2$

$25-x^2-y^2=4$

$21 = x^2+y^2$

$\sqrt{25-x^2-y^2} = 3$

$25-x^2-y^2=9$

$16 = x^2+y^2$

$\sqrt{25-x^2-y^2} = 4$

$25-x^2-y^2=16$

$9 = x^2+y^2$

18) $f(x,y) = \sqrt{y-x}$

a) Domain: set of all (x,y) such that $y-x \geq 0 \Rightarrow y \geq x$

b) Range: $f(x,y) = z \geq 0$

c) level curves are straight lines of the form $y-x=c$ where $c \geq 0$

d) boundary is $\sqrt{y-x} = 0 \Rightarrow y = x$, a straight line

e) closed

f) unbounded

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20) $f(x,y) = x^2 - y^2$

a) Domain: all points in the xy-plane

b) Range: all real numbers

c) level curves: for $f(x,y)=0$, the union of the lines $y = \pm x$
for $f(x,y) = c \neq 0$, hyperbolas centered at $(0,0)$ with foci on the x-axis if $c>0$ and on the y-axis if $c<0$

d) no boundary points

e) both open and closed

f) unbounded

22) $f(x,y) = \frac{y}{x^2}$

a) Domain: set of all (x,y) such that $(x,y) \neq (0,y)$

b) Range: all real numbers

c) level curves: for $f(x,y)=0$, the x-axis minus the origin
for $f(x,y) = c \neq 0$, the parabolas $y = cx^2$ minus the origin

d) boundary is the line $x=0$

e) open

f) unbounded

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24) $f(x,y) = \sqrt{9-x^2-y^2}$

a) Domain: set of all (x,y) satisfying $9-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 9$

b) Range: $0 \leq z = f(x,y) \leq 3$

c) level curves are circles with center $(0,0)$ and radii $r \leq 3$

d) boundary is the circle $x^2+y^2=9$

e) closed

f) bounded

26) $f(x,y) = e^{-(x^2+y^2)}$

a) Domain: all points in the xy-plane

b) Range: $0 \le z = f(x,y) \le 1$

c) level curves are the origin itself and the circles with center $(0,0)$ and radii $r > 0$

d) no boundary points

e) both open and closed

f) unbounded

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28) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

a) Domain: set of all (x,y) such that $x \neq 0$

b) Range: $\frac{-\pi}{2} < z = f(x,y) < \frac{\pi}{2}$

c) level curves are the straight lines of the form $y = cx$, where c is any real number and $x \neq 0$.

d) boundary is the line $x = 0$

e) open

f) unbounded

30) $f(x,y) = \ln(9 - x^2 - y^2)$

a) Domain: set of all (x,y) such that $9 - x^2 - y^2 > 0 \Rightarrow x^2 + y^2 < 9$
inside the circle $x^2 + y^2 = 9$

b) Range: $f(x,y) = z < \ln 9$

c) level curves are circles with center $(0,0)$ with radii $r < 3$

d) boundary: the circle $x^2 + y^2 = 9$

e) open

f) bounded

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32) picture: e; equation: l $\quad z = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$

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34) picture: c; equation: k $\quad z = \dfrac{1}{4x^2 + y^2}$

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36) picture: b; equation: g $\quad z = \dfrac{-xy^2}{x^2 + y^2}$

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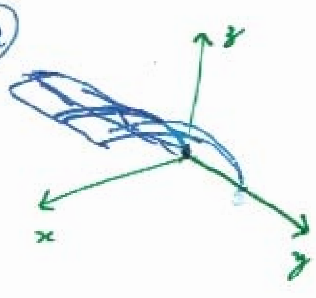
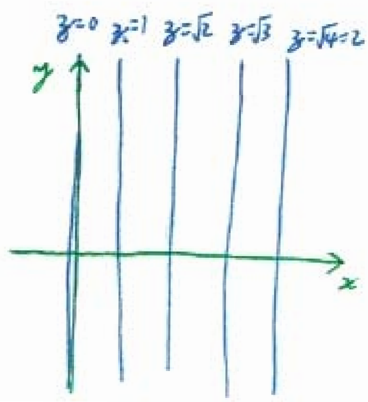
38) $f(x,y) = \sqrt{x}$

a

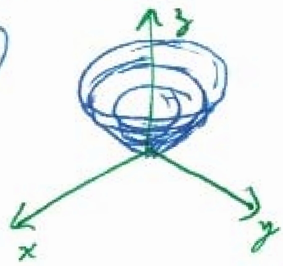


b



40) $f(x,y)=\sqrt{x^2+y^2}$

a



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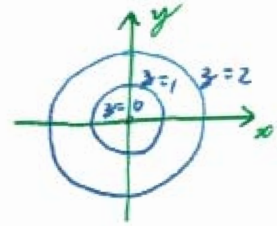
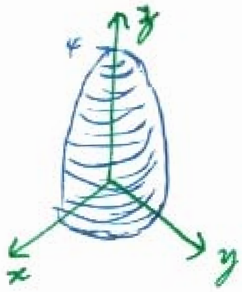


42) $f(x,y)=4-x^2-y^2$

a



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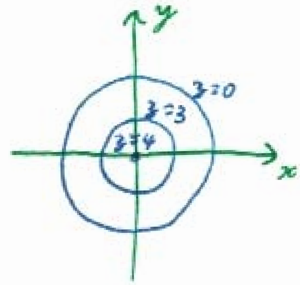
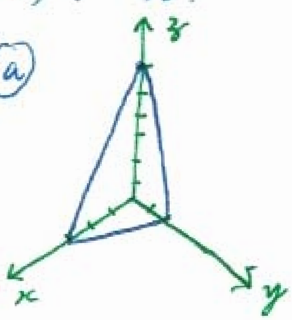


44) $f(x,y)=6-2x-3y$

a



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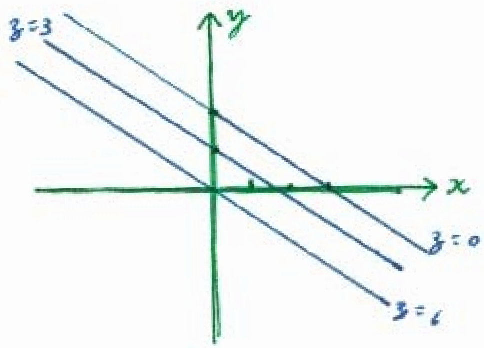
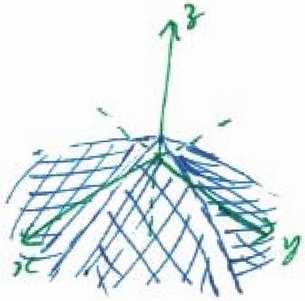


46) $f(x,y)=1-|x|-|y|$

a



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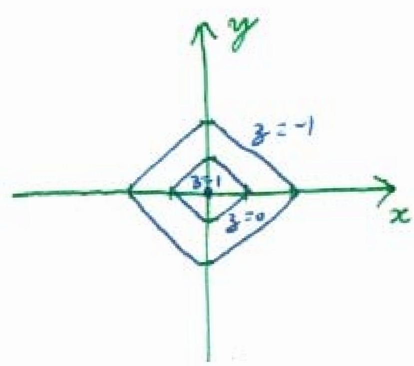
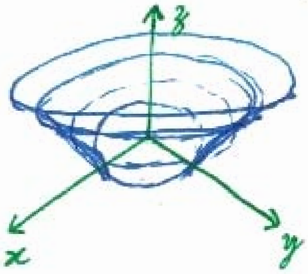
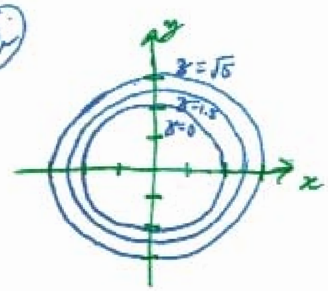


48) $f(x,y) = \sqrt{x^2+y^2-4}$

ⓐ



ⓑ



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50) $f(x,y) = \sqrt{x^2-1}$, $(1,0)$

$z = \sqrt{(1)^2-1} = \sqrt{0} = 0$

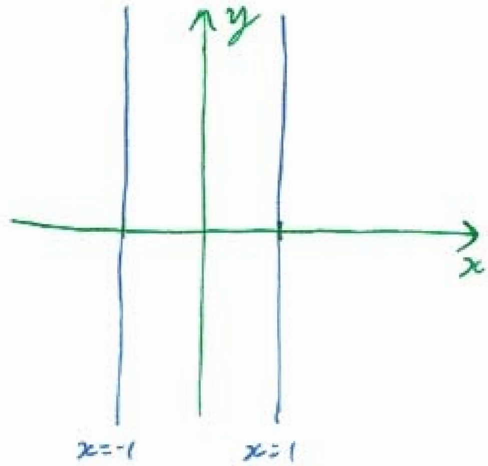
$\sqrt{x^2-1} = 0$

$x^2-1=0$

$(x+1)(x-1)=0$

$x+1=0 \mid x-1=0$

$x=-1 \mid x=1$



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52) $f(x,y) = \dfrac{2y-x}{x+y+1}$, $(-1,1)$

$z = \dfrac{2(1)-(-1)}{(-1)+(1)+1} = \dfrac{2+1}{1} = 3$

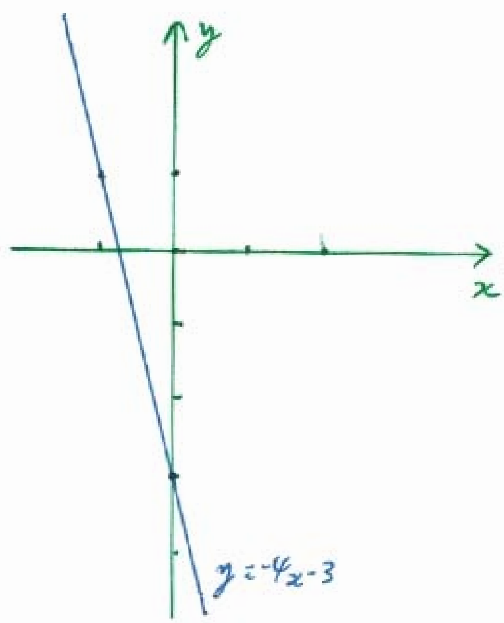
$\dfrac{2y-x}{x+y+1} = 3$

$2y - x = 3(x+y+1)$

$2y - x = 3x + 3y + 3$

$-4x - 3 = y$

$y = -4x - 3$

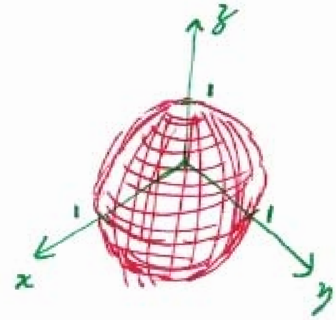


54) $f(x,y,z) = \ln(x^2+y^2+z^2)$

$\ln(x^2+y^2+z^2) = 0$

$\Downarrow$

$x^2+y^2+z^2 = 1$



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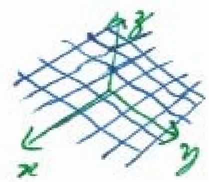
56) $f(x,y,z) = z$

$z = 1$



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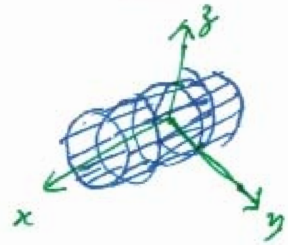
58) $f(x,y,z) = y^2+z^2$

$y^2+z^2 = 1$



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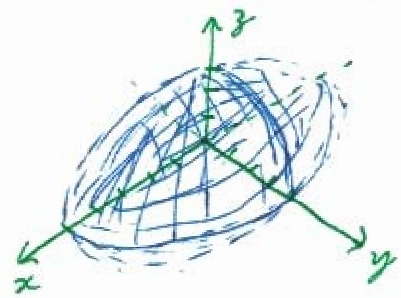
60) $f(x,y,z) = \dfrac{x^2}{25} + \dfrac{y^2}{16} + \dfrac{z^2}{9}$

$$\frac{x^2}{25} + \frac{y^2}{16} + \frac{z^2}{9} = 1$$



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62) $f(x,y,z) = \ln(x^2+y+z^2)$; $(-1,2,1)$

$w = f(x,y,z)$

$w = \ln((-1)^2+(2)+(1)^2) = \ln(1+2+1) = \ln 4 \Rightarrow$

$\ln 4 = \ln(x^2+y+z^2)$

$4 = x^2+y+z^2$

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64) $g(x,y,z) = \dfrac{x-y+z}{2x+y-z}$; $(1,0,-2)$

$w = \dfrac{x-y+z}{2x+y-z}$

$w = \dfrac{(1)-(0)+(-2)}{2(1)+(0)-(-2)} = \dfrac{1-2}{2+2} = \dfrac{-1}{4} \Rightarrow$

$\dfrac{-1}{4} = \dfrac{x-y+z}{2x+y-z}$

$-1(2x+y-z) = 4(x-y+z) \Rightarrow$

$-2x-y+z = 4x-4y+4z$

$0 = 6x-3y+3z$

$0 = 3(2x-y+z)$

$0 = 2x-y+z$