Lines and Line Segments in Space

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ or $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\mathbf{v} = \vec{v}$ (see Figure in the text, Figure 12.36 in 14th edition). Thus, $\overrightarrow{P_0P} = t\mathbf{v} = t\vec{v}$ for some scalar parameter t. The value of t depends on the location of the point P along the line, and the domain of t is $(-\infty, \infty)$. The expanded form of the equation $\overrightarrow{P_0P} = t\mathbf{v} = t\vec{v}$ is

$$(x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k} = t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$
$$(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} = t(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}),$$

which can be rewritten as

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k} + t(v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$
$$x\vec{i} + y\vec{j} + z\vec{k} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} + t(v_1\vec{i} + v_2\vec{j} + v_3\vec{k}). \qquad (1)$$

If $\mathbf{r}(t) = \vec{r}(t)$ is the position vector of a point $P(x, y, z)$ on the line and $\mathbf{r}_0 = \vec{r}_0$ is the position vector of the point $P_0(x_0, y_0, z_0)$, then Equation (1) gives the following vector form for the equation of a line in space.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = \vec{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad \vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty, \qquad (2)$$

where $\mathbf{r} = \vec{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0 = \vec{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Parametric Equation for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ or $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \qquad (3)$$

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} = \mathbf{r}_0 + t|\mathbf{v}|\frac{\mathbf{v}}{|\mathbf{v}|} = \underset{\substack{\textit{Initial}\\ \textit{position}}}{\mathbf{r}_0} + \underset{\textit{Time}}{t}\ \underset{\textit{Speed}}{|\mathbf{v}|}\ \underset{\textit{Direction}}{\frac{\mathbf{v}}{|\mathbf{v}|}}$$
$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \vec{r}(t) = \vec{r}_0 + t\vec{v} = \underset{\substack{\textit{Initial}\\ \textit{position}}}{\vec{r}_0} + \underset{\textit{Time}}{t}\ \underset{\textit{Speed}}{|\vec{v}|}\ \underset{\textit{Direction}}{\frac{\vec{v}}{|\vec{v}|}} \qquad (4)$$

The Distance from a Point to a Line in Space

Distance from a Point S to a Line Through P Parallel to $\mathbf{v} = \vec{v}$

$$d = \frac{\left|\overrightarrow{PS} \times \mathbf{v}\right|}{|\mathbf{v}|} = \frac{\left|\overrightarrow{PS} \times \vec{v}\right|}{|\vec{v}|} \qquad (5)$$

An Equation for a Plane in Space

Equation for a Plane

The plane trough $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ or $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ has

Vector equation: $\mathbf{n} \bullet \overrightarrow{P_0P} = 0 \quad \vec{n} \bullet \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$ where $D = Ax_0 + By_0 + Cz_0$

The vector $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ or $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ is normal to the plane $Ax + By + Cz = D$.

Distance from a Point S to a Plane with Normal $\mathbf{n} = \vec{n}$ at Point P

$$d = \left|\overrightarrow{PS} \bullet \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\overrightarrow{PS} \bullet \frac{\vec{n}}{|\vec{n}|}\right| \qquad (6)$$

MATH 21200 section 12.5

2) $P(1,2,-1)$ $Q(-1,0,1)$

$\overrightarrow{PQ} = \{(-1)-(1)\}\vec{i} + \{(0)-(2)\}\vec{j} + \{(1)-(-1)\}\vec{k} = -2\vec{i} - 2\vec{j} + 2\vec{k}$

$x=(1)+(-2)t$ $y=(2)+(-2)t$ $z=(-1)+(2)t$

$x=1-2t$ $y=2-2t$ $z=-1+2t$

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4) $P(1,2,0)$ $Q(1,1,-1)$

$\overrightarrow{PQ} = \{(1)-(1)\}\vec{i} + \{(1)-(2)\}\vec{j} + \{(-1)-(0)\}\vec{k} = 0\vec{i} - 1\vec{j} - 1\vec{k}$
$= -\vec{j} - \vec{k}$

$x=(1)+(0)t$ $y=(2)+(-1)t$ $z=(0)+(-1)t$

$x=1$ $y=2-t$ $z=-t$

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6) $(3,-2,1)$ $\parallel$ to $x=1+2t$, $y=2-t$, $z=3t$
$\parallel$ to $2\vec{i} - 1\vec{j} + 3\vec{k}$

$x=(3)+(2)t$ $y=(-2)+(-1)t$ $z=(1)+(3)t$

$x=3+2t$ $y=-2-t$ $z=1+3t$

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8) $(2,4,5)$ $\perp$ to plane $3x+7y-5z=21 \Rightarrow \vec{n} = 3\vec{i} + 7\vec{j} - 5\vec{k}$

$x=(2)+(3)t$ $y=(4)+(7)t$ $z=(5)+(-5)t$

$x=2+3t$ $y=4+7t$ $z=5-5t$

10) $(2,3,0)$ $\perp$ to $\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{v} = 3\vec{i} + 4\vec{j} + 5\vec{k}$

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = +\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}\vec{k}$$

$$= \{(2)(5)-(3)(4)\}\vec{i} - \{(1)(5)-(3)(3)\}\vec{j} + \{(1)(4)-(2)(3)\}\vec{k}$$

$$= \{10-12\}\vec{i} - \{5-9\}\vec{j} + \{4-6\}\vec{k} = \{-2\}\vec{i} - \{-4\}\vec{j} + \{-2\}\vec{k}$$

$$= -2\vec{i} + 4\vec{j} - 2\vec{k}$$

$x = (2) + (-2)t$ $\quad$ $y = (3) + (4)t$ $\quad$ $z = (0) + (-2)t$

$x = 2 - 2t$ $\quad$ $y = 3 + 4t$ $\quad$ $z = -2t$

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12) the z-axis $\Rightarrow$ vector: $0\vec{i} + 0\vec{j} + 1\vec{k} = \vec{k}$ point: $(0,0,0)$

$x = (0) + (0)t$ $\quad$ $y = (0) + (0)t$ $\quad$ $z = (0) + (1)t$

$x = 0$ $\quad$ $y = 0$ $\quad$ $z = t$

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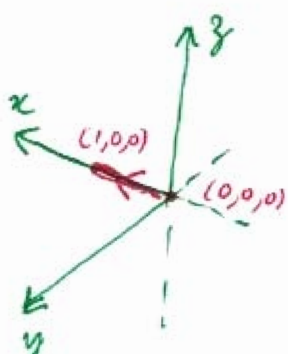
14) $P(0,0,0)$ $Q(1,0,0)$

$$\overrightarrow{PQ} = \{(1)-(0)\}\vec{i} + \{(0)-(0)\}\vec{j} + \{(0)-(0)\}\vec{k}$$

$$= 1\vec{i} + 0\vec{j} + 0\vec{k} = \vec{i}$$



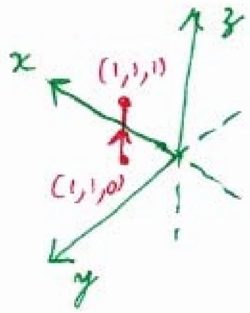
$(0,0,0)$

$x = (0) + (1)t$ $\quad$ $y = (0) + (0)t$ $\quad$ $z = (0) + (0)t$

$x = t$ $\quad$ $y = 0$ $\quad$ $z = 0$

$0 \le t \le 1$

16) $P(1,1,0) \quad Q(1,1,1)$



$$\overrightarrow{PQ} = \{(1)-(1)\}\vec{i} + \{(1)-(1)\}\vec{j} + \{(1)-(0)\}\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + 1\vec{k} = \vec{k}$$

$(1,1,0)$

$x = (1)+(0)t \qquad y = (1)+(0)t \qquad z = (0)+(1)t$

$x = 1 \qquad y = 1 \qquad z = t$

$0 \le t \le 1$

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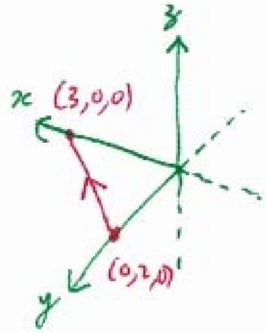
18) $P(0,2,0) \quad Q(3,0,0)$



$$\overrightarrow{PQ} = \{(3)-(0)\}\vec{i} + \{(0)-(2)\}\vec{j} + \{(0)-(0)\}\vec{k}$$

$$= 3\vec{i} - 2\vec{j} + 0\vec{k} = 3\vec{i} - 2\vec{j}$$

$(0,2,0)$

$x = (0)+(3)t \qquad y = (2)+(-2)t \qquad z = (0)+(0)t$

$x = 3t \qquad y = 2-2t \qquad z = 0$

$0 \le t \le 1$

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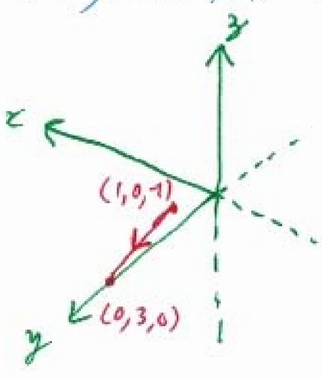
20) $P(1,0,-1) \quad Q(0,3,0)$



$$\overrightarrow{PQ} = \{(0)-(1)\}\vec{i} + \{(3)-(0)\}\vec{j} + \{(0)-(-1)\}\vec{k}$$

$$= -1\vec{i} + 3\vec{j} + 1\vec{k} = -\vec{i} + 3\vec{j} + \vec{k}$$

$(1,0,-1)$

$x = (1)+(-1)t \qquad y = (0)+(3)t \qquad z = (-1)+(1)t$

$x = 1-t \qquad y = 3t \qquad z = -1+t$

$0 \le t \le 1$

22) $(1,-1,3)$ $\parallel$ to $3x+y+z=7$

$3(x-(1))+1(y-(-1))+1(z-(3))=0$

$3(x-1)+(y+1)+(z-3)=0$

$3x-3+y+1+z-3=0$

$3x+y+z-5=0$

$3x+y+z=5$

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24) $P(2,4,5)$ $Q(1,5,7)$ $R(-1,6,8)$

$\vec{PQ}=\{(1)-(2)\}\vec{i}+\{(5)-(4)\}\vec{j}+\{(7)-(5)\}\vec{k}=-1\vec{i}+1\vec{j}+2\vec{k}=-\vec{i}+\vec{j}+2\vec{k}$

$\vec{PR}=\{(-1)-(2)\}\vec{i}+\{(6)-(4)\}\vec{j}+\{(8)-(5)\}\vec{k}=-3\vec{i}+2\vec{j}+3\vec{k}$

$$\vec{n}=\vec{PQ}\times\vec{PR}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 1 & 2\\ -3 & 2 & 3\end{vmatrix}=+\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix}\vec{i}-\begin{vmatrix}-1 & 2\\ -3 & 3\end{vmatrix}\vec{j}+\begin{vmatrix}-1 & 1\\ -3 & 2\end{vmatrix}\vec{k}$$

$=\{(1)(3)-(2)(2)\}\vec{i}-\{(-1)(3)-(2)(-3)\}\vec{j}+\{(-1)(2)-(1)(-3)\}\vec{k}$

$=\{3-4\}\vec{i}-\{-3+6\}\vec{j}+\{-2+3\}\vec{k}=\{-1\}\vec{i}-\{3\}\vec{j}+\{1\}\vec{k}=-1\vec{i}-3\vec{j}+1\vec{k}$

$=-\vec{i}-3\vec{j}+\vec{k}$

now use any point

$(-1)(x-(2))+(-3)(y-(4))+(1)(z-(5))=0$

$-(x-2)-3(y-4)+(z-5)=0$

$-x+2-3y+12+z-5=0$

$-x-3y+z+9=0$

$9=x+3y-z \Longrightarrow x+3y-z=9$

26) $A(1,-2,1)$ $\quad O(0,0,0)$

$\vec{OA} = \{(1)-(0)\}\vec{i} + \{(-2)-(0)\}\vec{j} + \{(1)-(0)\}\vec{k} = 1\vec{i} - 2\vec{j} + 1\vec{k}$

$= \vec{i} - 2\vec{j} + \vec{k}$

$(1)(x-(1)) + (-2)(y-(-2)) + (1)(z-(1)) = 0$

$(x-1) - 2(y+2) + (z-1) = 0$

$x - 1 - 2y - 4 + z - 1 = 0$

$x - 2y + z - 6 = 0$

$x - 2y + z = 6$

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28) $x = t \quad y = -t+2 \quad z = t+1$

$x = 2s+2 \quad y = s+3 \quad z = 5s+6$

$t = x = 2s+2 \quad -t+2 = y = s+3$

$t = 2s+2 \quad -t+2 = s+3$

$-(2s+2)+2 = s+3$

$-2s-2+2 = s+3$

$-2s = s+3$

$-3 = 3s$

$-1 = s$

$t = 2(-1)+2$

$t = -2+2$

$t = 0$

the lines intersect when $t=0$ and $s=-1$

so the point is $x=(0)=0$, $y=-(0)+2=2$, $z=(0)+1=1$

$P(0,2,1)$

$\vec{n}_1 = 1\vec{i} - 1\vec{j} + 1\vec{k} \quad n_2 = 2\vec{i} + 1\vec{j} + 5\vec{k}$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = +\begin{vmatrix} -1 & 1 \\ 1 & 5 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\vec{k}$$

$= \{(-1)(5)-(1)(1)\}\vec{i} - \{(1)(5)-(1)(2)\}\vec{j} + \{(1)(1)-(-1)(2)\}\vec{k} = -6\vec{i} - 3\vec{j} + 3\vec{k}$

plane: $(-6)(x-(0)) + (-3)(y-(2)) + (3)(z-(1)) = 0$

$-6(x) - 3(y-2) + 3(z-1) = 0$

$-6x - 3y + 6 + 3z - 3 = 0$

$-6x - 3y + 3z + 3 = 0$

$3 = 6x + 3y - 3z$

$6x + 3y - 3z = 3$

30) L1: $x = t,\ y = 3 - 3t,\ z = -2 - t\ ;\ -\infty < t < \infty$

L2: $x = 1 + s,\ y = 4 + s,\ z = -1 + s\ ;\ -\infty < s < \infty$

$\vec{v}_1 = 1\vec{i} - 3\vec{j} - 1\vec{k}$ $\qquad$ $\vec{v}_2 = 1\vec{i} + 1\vec{j} + 1\vec{k}$

point of intersection:

$t = 1 + s$ $\qquad$ $-2 - t = -1 + s$ $\qquad$ $x = (0) = 0$

$-2 - (1 + s) = -1 + s$ $\qquad$ $y = 3 - 3(0) = 3$ $\qquad$ $P(0, 3, -2)$

$-2 - 1 - s = -1 + s$ $\qquad$ $z = -2 - (0) = -2$

$t = 1 + (-1) = 0$ $\qquad$ $-2 = 2s$

$-1 = s$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = +\begin{vmatrix} -3 & -1 \\ 1 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix}\vec{k}$$

$$= \{(-3)(1) - (-1)(1)\}\vec{i} - \{(1)(1) - (-1)(1)\}\vec{j} + \{(1)(1) - (-3)(1)\}\vec{k}$$

$$= \{-3 + 1\}\vec{i} - \{1 + 1\}\vec{j} + \{1 + 3\}\vec{k} = -2\vec{i} - 2\vec{j} + 4\vec{k}$$

plane:

$$(-2)(x - (0)) + (-2)(y - (3)) + (4)(z - (-2)) = 0$$

$$-2x - 2(y - 3) + 4(z + 2) = 0$$

$$-2x - 2y + 6 + 4z + 8 = 0$$

$$-2x - 2y + 4z + 14 = 0$$

$$14 = 2x + 2y - 4z$$

$2x + 2y - 4z = 14$

or

$x + y - 2z = 7$

38) point: line: $x = 10 + 4t$, $y = -3$, $z = 4t$

$S(-1,4,3)$ $P(10,-3,0)$ and $\vec{v} = 4\vec{i} + 0\vec{j} + 4\vec{k} = 4\vec{i} + 4\vec{k}$

$$\vec{PS} = \{(-1)-(10)\}\vec{i} + \{(4)-(-3)\}\vec{j} + \{(3)-(0)\}\vec{k} = -11\vec{i} + 7\vec{j} + 3\vec{k}$$

$$\vec{PS} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -11 & 7 & 3 \\ 4 & 0 & 4 \end{vmatrix} = +\begin{vmatrix} 7 & 3 \\ 0 & 4 \end{vmatrix}\vec{i} - \begin{vmatrix} -11 & 3 \\ 4 & 4 \end{vmatrix}\vec{j} + \begin{vmatrix} -11 & 7 \\ 4 & 0 \end{vmatrix}\vec{k}$$

$$= \{(7)(4)-(3)(0)\}\vec{i} - \{(-11)(4)-(3)(4)\}\vec{j} + \{(-11)(0)-(7)(4)\}\vec{k}$$

$$= \{28-0\}\vec{i} - \{-44-12\}\vec{j} + \{0-28\}\vec{k} = 28\vec{i} + 56\vec{j} - 28\vec{k}$$

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{(28)^2 + (56)^2 + (28)^2}}{\sqrt{(4)^2 + (0)^2 + (4)^2}} = \frac{\sqrt{(28)^2\{(1)^2 + (2)^2 + (1)^2\}}}{\sqrt{(4)^2\{(1)^2 + (1)^2\}}} = \frac{28\sqrt{6}}{4\sqrt{2}}$$

$$= \frac{28}{4}\sqrt{\frac{6}{2}} = 7\sqrt{3}$$

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40) point: plane: $3x + 2y + 6z = 6 \Rightarrow \vec{n} = 3\vec{i} + 2\vec{j} + 6\vec{k}$

$S(0,0,0)$ by trial & error: $P(0,0,1)$ is on this plane

$$\vec{PS} = \{(0)-(0)\}\vec{i} + \{(0)-(0)\}\vec{j} + \{(0)-(1)\}\vec{k} = 0\vec{i} + 0\vec{j} - 1\vec{k} = -\vec{k}$$

$$\frac{\vec{n}}{|\vec{n}|} = \frac{3\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{(3)^2 + (2)^2 + (6)^2}} = \frac{3\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{9 + 4 + 36}} = \frac{3\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{49}} = \frac{3}{\sqrt{49}}\vec{i} + \frac{2}{\sqrt{49}}\vec{j} + \frac{6}{\sqrt{49}}\vec{k}$$

$$d = \left|\vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|}\right| = \left|(0)\left(\frac{3}{\sqrt{49}}\right) + (0)\left(\frac{2}{\sqrt{49}}\right) + (-1)\left(\frac{6}{\sqrt{49}}\right)\right| = \left|\frac{-6}{\sqrt{49}}\right| = \frac{6}{\sqrt{49}}$$

32) $P_1(1,2,3)$ $P_2(3,2,1)$ $\perp$ to $4x-y+2z=7$

$\vec{n}=4\vec{i}-1\vec{j}+2\vec{k}$

$$\overrightarrow{P_1P_2}=\{(3)-(1)\}\vec{i}+\{(2)-(2)\}\vec{j}+\{(1)-(3)\}\vec{k}=2\vec{i}+0\vec{j}-2\vec{k}=2\vec{i}-2\vec{k}$$

$$\overrightarrow{P_1P_2}\times\vec{n}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 0 & -2\\ 4 & -1 & 2\end{vmatrix}=+\begin{vmatrix}0 & -2\\ -1 & 2\end{vmatrix}\vec{i}-\begin{vmatrix}2 & -2\\ 4 & 2\end{vmatrix}\vec{j}+\begin{vmatrix}2 & 0\\ 4 & -1\end{vmatrix}\vec{k}$$

$$=\{(0)(2)-(-2)(-1)\}\vec{i}-\{(2)(2)-(-2)(4)\}\vec{j}+\{(2)(-1)-(0)(4)\}\vec{k}$$

$$=-2\vec{i}-12\vec{j}-2\vec{k}$$

using the point $P_1(1,2,3)$

$$(-2)(x-(1))+(-12)(y-(2))+(-2)(z-(3))=0$$

$$-2(x-1)-12(y-2)-2(z-3)=0$$

$$-2x+2-12y+24-2z+6=0$$

$$-2x-12y-2z+32=0$$

$$32=2x+12y+2z$$

$$2x+12y+2z=32$$

or

$$x+6y+z=16$$

34) point: $\quad$ line: $x = 5+3t,\ y = 5+4t,\ z = -3-5t$

$S(0,0,0) \qquad P(5,5,-3)$ and $\vec{v} = 3\vec{i} + 4\vec{j} - 5\vec{k}$

$$\overrightarrow{PS} = \{(0)-(5)\}\vec{i} + \{(0)-(5)\}\vec{j} + \{(0)-(-3)\}\vec{k} = -5\vec{i} - 5\vec{j} + 3\vec{k}$$

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & -5 & 3 \\ 3 & 4 & -5 \end{vmatrix} = +\begin{vmatrix} -5 & 3 \\ 4 & -5 \end{vmatrix}\vec{i} - \begin{vmatrix} -5 & 3 \\ 3 & -5 \end{vmatrix}\vec{j} + \begin{vmatrix} -5 & -5 \\ 3 & 4 \end{vmatrix}\vec{k}$$

$$= \{(-5)(-5) - (3)(4)\}\vec{i} - \{(-5)(-5) - (3)(3)\}\vec{j} + \{(-5)(4) - (-5)(3)\}\vec{k}$$

$$= \{25 - 12\}\vec{i} - \{25 - 9\}\vec{j} + \{-20 + 15\}\vec{k} = 13\vec{i} - 16\vec{j} - 5\vec{k}$$

$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{(13)^2 + (-16)^2 + (-5)^2}}{\sqrt{(3)^2 + (4)^2 + (5)^2}} = \frac{\sqrt{169 + 256 + 25}}{\sqrt{9 + 16 + 25}} = \frac{\sqrt{450}}{\sqrt{50}} = \sqrt{\frac{450}{50}} = \sqrt{9} = 3$$

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36) point: $\quad$ line: $x = 2t,\ y = 1 + 2t,\ z = 2t$

$S(2,1,-1) \qquad P(0,1,0)$ and $\vec{v} = 2\vec{i} + 2\vec{j} + 2\vec{k}$

$$\overrightarrow{PS} = \{(2)-(0)\}\vec{i} + \{(1)-(1)\}\vec{j} + \{(-1)-(0)\}\vec{k} = 2\vec{i} + 0\vec{j} - 1\vec{k} = 2\vec{i} - \vec{k}$$

$$\overrightarrow{PS} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = +\begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 0 \\ 2 & 2 \end{vmatrix}\vec{k}$$

$$= \{(0)(2) - (-1)(2)\}\vec{i} - \{(2)(2) - (-1)(2)\}\vec{j} + \{(2)(2) - (0)(2)\}\vec{k} = 2\vec{i} - 6\vec{j} + 4\vec{k}$$

$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{(2)^2 + (-6)^2 + (4)^2}}{\sqrt{(2)^2 + (2)^2 + (2)^2}} = \frac{\sqrt{4 + 36 + 16}}{\sqrt{4 + 4 + 4}} = \frac{\sqrt{4(1 + 9 + 4)}}{\sqrt{4(1 + 1 + 1)}} = \frac{\sqrt{4}\sqrt{1 + 9 + 4}}{\sqrt{4}\sqrt{1 + 1 + 1}}$$

$$= \frac{\sqrt{14}}{\sqrt{3}} = \sqrt{\frac{14}{3}}$$

42) point: $S(2,2,3)$

plane: $2x+y+2z=4 \Rightarrow \vec{n}=2\vec{i}+\vec{j}+2\vec{k}$

by trial & error: $P(2,0,0)$ is on this plane

$$\vec{PS}=\{(2)-(2)\}\vec{i}+\{(2)-(0)\}\vec{j}+\{(3)-(0)\}\vec{k}=0\vec{i}+2\vec{j}+3\vec{k}=2\vec{j}+3\vec{k}$$

$$|\vec{n}|=\sqrt{(2)^2+(1)^2+(2)^2}=\sqrt{4+1+4}=\sqrt{9}=3$$

$$\frac{\vec{n}}{|\vec{n}|}=\frac{2\vec{i}+\vec{j}+2\vec{k}}{3}=\frac{2}{3}\vec{i}+\frac{1}{3}\vec{j}+\frac{2}{3}\vec{k}$$

$$d=\left|\vec{PS}\cdot\frac{\vec{n}}{|\vec{n}|}\right|=\left|(0)\left(\tfrac{2}{3}\right)+(2)\left(\tfrac{1}{3}\right)+(3)\left(\tfrac{2}{3}\right)\right|=\left|\frac{2}{3}+\frac{6}{3}\right|=\left|\frac{8}{3}\right|=\frac{8}{3}$$

---

44) point: $S(1,0,-1)$

plane: $-4x+y+z=4 \Rightarrow \vec{n}=-4\vec{i}+\vec{j}+\vec{k}$

by trial & error: $P(-1,0,0)$ is on this plane

$$\vec{PS}=\{(1)-(-1)\}\vec{i}+\{(0)-(0)\}\vec{j}+\{(-1)-(0)\}\vec{k}=2\vec{i}+0\vec{j}-1\vec{k}=2\vec{i}-\vec{k}$$

$$|\vec{n}|=\sqrt{(-4)^2+(1)^2+(1)^2}=\sqrt{16+1+1}=\sqrt{18}=3\sqrt{2} \qquad \frac{\vec{n}}{|\vec{n}|}=\frac{-4\vec{i}+\vec{j}+\vec{k}}{3\sqrt{2}}=\frac{-4}{3\sqrt{2}}\vec{i}+\frac{1}{3\sqrt{2}}\vec{j}+\frac{1}{3\sqrt{2}}\vec{k}$$

$$d=\left|\vec{PS}\cdot\frac{\vec{n}}{|\vec{n}|}\right|=\left|(2)\left(\frac{-4}{3\sqrt{2}}\right)+(0)\left(\frac{1}{3\sqrt{2}}\right)+(-1)\left(\frac{1}{3\sqrt{2}}\right)\right|=\left|\frac{-8}{3\sqrt{2}}+0-\frac{1}{3\sqrt{2}}\right|=\left|\frac{-9}{3\sqrt{2}}\right|=\left|\frac{-3}{\sqrt{2}}\right|=\frac{3}{\sqrt{2}}$$

---

46) line: $x=2+t,\ y=1+t,\ z=\frac{-1}{2}-\frac{1}{2}t$

let $t=-1$ then we get a point on this line $S(1,0,0)$

"this will be a point that we end up with simpler calculations"

$$\vec{v}=1\vec{i}+1\vec{j}-\tfrac{1}{2}\vec{k}=\vec{i}+\vec{j}-\tfrac{1}{2}\vec{k}$$

plane: $x+2y+6z=1 \Rightarrow \vec{n}=1\vec{i}+2\vec{j}+6\vec{k}$

46) continued...

$\vec{v} \cdot \vec{n} = (1)(1) + (1)(2) + (\frac{-1}{2})(6) = 1 + 2 - 3 = 0$ this implies that the line is parallel to the plane.

to make our calculation easier, we can use $P(10,0,0)$ which is on the plane

$\vec{PS} = \{(1)-(10)\}\vec{i} + \{(0)-(0)\}\vec{j} + \{(0)-(0)\}\vec{k} = -9\vec{i} + 0\vec{j} + 0\vec{k} = -9\vec{i}$

$|\vec{n}| = \sqrt{(1)^2 + (2)^2 + (6)^2} = \sqrt{1+4+36} = \sqrt{41}$ $\quad \frac{\vec{n}}{|\vec{n}|} = \frac{1\vec{i} + 2\vec{j} + 6\vec{k}}{\sqrt{41}} = \frac{1}{\sqrt{41}}\vec{i} + \frac{2}{\sqrt{41}}\vec{j} + \frac{6}{\sqrt{41}}\vec{k}$

$$d = \left|\vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|}\right| = \left|(-9)\left(\frac{1}{\sqrt{41}}\right) + (0)\left(\frac{2}{\sqrt{41}}\right) + (0)\left(\frac{6}{\sqrt{41}}\right)\right| = \left|\frac{-9}{\sqrt{41}}\right| = \frac{9}{\sqrt{41}}$$

---

48) $5x + y - z = 10 \Longrightarrow \vec{n}_1 = 5\vec{i} + 1\vec{j} - 1\vec{k}$

$|\vec{n}_1| = \sqrt{(5)^2 + (1)^2 + (-1)^2} = \sqrt{25+1+1} = \sqrt{27}$

$x - 2y + 3z = -1 \Longrightarrow \vec{n}_2 = 1\vec{i} - 2\vec{j} + 3\vec{k}$

$|\vec{n}_2| = \sqrt{(1)^2 + (-2)^2 + (3)^2} = \sqrt{1+4+9} = \sqrt{14}$

$\vec{n}_1 \cdot \vec{n}_2 = (5)(1) + (1)(-2) + (-1)(3) = 5 - 2 - 3 = 0$

$$\theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right) = \cos^{-1}\left(\frac{(0)}{(\sqrt{27})(\sqrt{14})}\right) = \cos^{-1}(0) = \frac{\pi}{2}$$

52) line: $x=2$, $y=3+2t$, $z=1-2t$

$\vec{v}=0\vec{i}+2\vec{j}-2\vec{k}=2\vec{j}-2\vec{k}$

$|\vec{v}|=\sqrt{(0)^2+(2)^2+(-2)^2}=\sqrt{0+4+4}=\sqrt{4(1+1)}=2\sqrt{2}$

plane: $x-y+z=0 \Longrightarrow \vec{n}=1\vec{i}-1\vec{j}+1\vec{k}$

$|\vec{n}|=\sqrt{(1)^2+(-1)^2+(1)^2}=\sqrt{1+1+1}=\sqrt{3}$

$\vec{v}\cdot\vec{n}=(0)(1)+(2)(-1)+(-2)(1)=0-2-2=-4$

$\theta=\cos^{-1}\left(\frac{\vec{v}\cdot\vec{n}}{|\vec{v}||\vec{n}|}\right)=\cos^{-1}\left(\frac{(-4)}{(2\sqrt{2})(\sqrt{3})}\right)=\underline{\underline{\cos^{-1}\left(\frac{-2}{\sqrt{6}}\right)}}$

---

58) line: $x=2$, $y=3+2t$, $z=-2-2t$

plane: $6x+3y-4z=-12$

$6(2)+3(3+2t)-4(-2-2t)=-12$

$12+9+6t+8+8t=-12$

$14t+29=-12$

$14t=-41$

$t=\frac{-41}{14}$

$x=2$

$y=3+2\left(\frac{-41}{14}\right)=3-\frac{41}{7}$

$=\frac{21}{7}-\frac{41}{7}=\frac{-20}{7}$

$z=-2-2\left(\frac{-41}{14}\right)=-2+\frac{41}{7}$

$=\frac{-14}{7}+\frac{41}{7}=\frac{27}{7}$

the point is $\left(2, \frac{-20}{7}, \frac{27}{7}\right)$

60) line: $x = -1 + 3t$, $y = -2$, $z = 5t$

plane: $2x - 3z = 7$

$2(-1+3t) - 3(5t) = 7$

$-2 + 6t - 15t = 7$

$-2 - 9t = 7$

$-9 = 9t$

$-1 = t$

$x = -1 + 3(-1) = -4$

$y = -2$

$z = 5(-1) = -5$

The point is $(-4, -2, -5)$

---

62) $3x - 6y - 2z = 3 \implies \vec{n}_1 = 3\vec{i} - 6\vec{j} - 2\vec{k}$

$2x + y - 2z = 2 \implies \vec{n}_2 = 2\vec{i} + 1\vec{j} - 2\vec{k}$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = +\begin{vmatrix} -6 & -2 \\ 1 & -2 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & -2 \\ 2 & -2 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & -6 \\ 2 & 1 \end{vmatrix}\vec{k}$$

$$= \{(-6)(-2) - (-2)(1)\}\vec{i} - \{(3)(-2) - (-2)(2)\}\vec{j} + \{(3)(1) - (-6)(2)\}\vec{k}$$

$$= \{12 + 2\}\vec{i} - \{-6 + 4\}\vec{j} + \{3 + 12\}\vec{k} = 14\vec{i} + 2\vec{j} + 15\vec{k}$$

by trial & error, the point $(1, 0, 0)$ is on both planes

parametrization:

$x = (1) + (14)t$ $\quad y = (0) + (2)t$ $\quad z = (0) + (15)t$

$x = 1 + 14t$ $\quad y = 2t$ $\quad z = 15t$

64) $5x-2y=11 \Rightarrow 5x-2y+0z=11 \Longrightarrow \vec{n}_1=5\vec{i}-2\vec{j}+0\vec{k}$

$4y-5z=-17 \Rightarrow 0x+4y-5z=-17 \Longrightarrow \vec{n}_2=0\vec{i}+4\vec{j}-5\vec{k}$

$$\vec{n}_1\times\vec{n}_2=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\5&-2&0\\0&4&-5\end{vmatrix}=+\begin{vmatrix}-2&0\\4&-5\end{vmatrix}\vec{i}-\begin{vmatrix}5&0\\0&-5\end{vmatrix}\vec{j}+\begin{vmatrix}5&-2\\0&4\end{vmatrix}\vec{k}$$

$$=\{(-2)(-5)-(0)(4)\}\vec{i}-\{(5)(-5)-(0)(0)\}\vec{j}+\{(5)(4)-(-2)(0)\}\vec{k}$$

$$=\{10-0\}\vec{i}-\{-25-0\}\vec{j}+\{20-0\}\vec{k}=10\vec{i}+25\vec{j}+20\vec{k}$$

by trial & error, the point $(1,-3,1)$ is on both planes

parametrization:

$x=(1)+(10)t \qquad y=(-3)+(25)t \qquad z=(1)+(20)t$

$x=1+10t \qquad y=-3+25t \qquad z=1+20t$

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68) $P_1(4,1,5)$; $\vec{n}_1=\vec{i}-2\vec{j}+\vec{k}$

$(1)(x-(4))+(-2)(y-(1))+(1)(z-(5))=0$

$(x-4)-2(y-1)+(z-5)=0$

$x-4-2y+2+z-5=0$

$x-2y+z-7=0$

$x-2y+z=7$

$P_2(3,-2,0)$; $\vec{n}_2-\sqrt{2}\vec{i}+2\sqrt{2}\vec{j}-\sqrt{2}\vec{k}$

$(-\sqrt{2})(x-(3))+(2\sqrt{2})(y-(-2))+(-\sqrt{2})(z-(0))=0$

$-\sqrt{2}(x-3)+2\sqrt{2}(y+2)-\sqrt{2}(z)=0$

$-\sqrt{2}x+3\sqrt{2}+2\sqrt{2}y+4\sqrt{2}-\sqrt{2}z=0$

$-\sqrt{2}x+2\sqrt{2}y-\sqrt{2}z+7\sqrt{2}=0$

$7\sqrt{2}=\sqrt{2}x-2\sqrt{2}y+\sqrt{2}z$

$\sqrt{2}x-2\sqrt{2}y+\sqrt{2}z=7\sqrt{2}$

72) $A_1 x + B_1 y + C_1 z = D_1 \Rightarrow \vec{n}_1 = A_1 \vec{i} + B_1 \vec{j} + C_1 \vec{k}$

$A_2 x + B_2 y + C_2 z = D_2 \Longrightarrow \vec{n}_2 = A_2 \vec{i} + B_2 \vec{j} + C_2 \vec{k}$

parallel: 1 if either vector $\vec{n}_1$ or $\vec{n}_2$ is a multiple of the other.

2 $\vec{n}_1 \times \vec{n}_2 = (A_1 \vec{i} + B_1 \vec{j} + C_1 \vec{k}) \times (A_2 \vec{i} + B_2 \vec{j} + C_2 \vec{k}) = \vec{0}$

perpendicular: when their normals are $\perp$

$\vec{n}_1 \cdot \vec{n}_2 = (A_1 \vec{i} + B_1 \vec{j} + C_1 \vec{k}) \cdot (A_2 \vec{i} + B_2 \vec{j} + C_2 \vec{k}) = 0$