Theorem 1 – Angle Between Two Vectors

The angle θ between two nonzero vectors $\mathbf{u} = \vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{u_1v_1 + u_2v_2 + u_3v_3}{|\vec{u}||\vec{v}|}\right).$$

Definition

The **dot product** $\mathbf{u}\bullet\mathbf{v} = \vec{u}\bullet\vec{v}$ ("$\mathbf{u} = \vec{u}$ **dot** $\mathbf{v} = \vec{v}$") of vectors $\mathbf{u} = \vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \vec{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$\mathbf{u}\bullet\mathbf{v} = \vec{u}\bullet\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$$

Dot Product and Angles

The angle between two nonzero vectors $\mathbf{u} = \vec{u}$ and $\mathbf{v} = \vec{v}$ is $\theta = \cos^{-1}\left(\frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{\vec{u}\bullet\vec{v}}{|\vec{u}||\vec{v}|}\right)$.

The dot product of two vectors $\mathbf{u} = \vec{u}$ and $\mathbf{v} = \vec{v}$ is given by $\mathbf{u}\bullet\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ or $\vec{u}\bullet\vec{v} = |\vec{u}||\vec{v}|\cos\theta$.

Orthogonal Vectors

Definition Vectors $\mathbf{u} = \vec{u}$ and $\mathbf{v} = \vec{v}$ are orthogonal if $\mathbf{u}\bullet\mathbf{v} = \vec{u}\bullet\vec{v} = 0$

Dot Product Properties and Vector Projections

Properties of the Dot Product

If $\mathbf{u} = \vec{u}$, $\mathbf{v} = \vec{v}$, $\mathbf{w} = \vec{w}$ be vectors and c a scalar, then

1. $\mathbf{u}\bullet\mathbf{v} = \mathbf{v}\bullet\mathbf{u}$ $\quad\quad$ $\vec{u}\bullet\vec{v} = \vec{v}\bullet\vec{u}$
2. $(c\mathbf{u})\bullet\mathbf{v} = \mathbf{u}\bullet(c\mathbf{v}) = c(\mathbf{u}\bullet\mathbf{v})$ $\quad\quad$ $(c\vec{u})\bullet\vec{v} = \vec{u}\bullet(c\vec{v}) = c(\vec{u}\bullet\vec{v})$
3. $\mathbf{u}\bullet(\mathbf{v}+\mathbf{w}) = \mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{w}$ $\quad\quad$ $\vec{u}\bullet(\vec{v}+\vec{w}) = \vec{u}\bullet\vec{v} + \vec{u}\bullet\vec{w}$
4. $\mathbf{u}\bullet\mathbf{u} = |\mathbf{u}|^2$ $\quad\quad$ $\vec{u}\bullet\vec{u} = |\vec{u}|^2$
5. $\mathbf{0}\bullet\mathbf{u} = 0$ $\quad\quad$ $\vec{0}\bullet\vec{u} = 0$

The vector projection of $\mathbf{u} = \vec{u}$ onto $\mathbf{v} = \vec{v}$ is the vector

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} \quad \text{or} \quad \text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\bullet\vec{v}}{|\vec{v}|^2}\right)v = \left(\frac{\vec{u}\bullet\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}$$

The scalar component of $\mathbf{u} = \vec{u}$ in the direction of $\mathbf{v} = \vec{v}$ is the scalar

$$|\mathbf{u}|\cos\theta = \frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{v}|} = \mathbf{u}\bullet\frac{\mathbf{v}}{|\mathbf{v}|} \quad \text{or} \quad |\vec{u}|\cos\theta = \frac{\vec{u}\bullet\vec{v}}{|\vec{v}|} = \vec{u}\bullet\frac{\vec{v}}{|\vec{v}|}$$

Work Definition

The **work** done by a constant force $\mathbf{F} = \vec{F}$ acting through a displacement $\mathbf{D} = \vec{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F}\bullet\mathbf{D} = |\mathbf{F}||\mathbf{D}|\cos\theta = \vec{F}\bullet\vec{D} = |\vec{F}||\vec{D}|\cos\theta$$

2) $\vec{v} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{k}$ $\qquad$ $\vec{u} = 5\vec{i} + 12\vec{j}$

$= \frac{3}{5}\vec{i} + 0\vec{j} + \frac{4}{5}\vec{k}$ $\qquad$ $= 5\vec{i} + 12\vec{j} + 0\vec{k}$

a) $\vec{v} \cdot \vec{u} = \left(\frac{3}{5}\right)(5) + (0)(12) + \left(\frac{4}{5}\right)(0) = 3 + 0 + 0 = 3$

$|\vec{v}| = \sqrt{\left(\frac{3}{5}\right)^2 + (0)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{(5)^2} + \frac{16}{(5)^2}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$

$|\vec{u}| = \sqrt{(5)^2 + (12)^2 + (0)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

b) $\cos\theta = \frac{\vec{v} \cdot \vec{u}}{|\vec{v}||\vec{u}|} = \frac{(3)}{(1)(13)} = \frac{3}{13}$

c) $|\vec{u}|\cos\theta = \frac{\vec{v} \cdot \vec{u}}{|\vec{v}|} = \frac{(3)}{(1)} = 3$

d) $\text{proj}_{\vec{v}}\,\vec{u} = \left(\frac{\vec{v} \cdot \vec{u}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|} = \left(\frac{(3)}{(1)}\right)\frac{\frac{3}{5}\vec{i} + \frac{4}{5}\vec{k}}{(1)}$

$= 3\left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{k}\right)$

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2*) $\vec{v} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$, $\vec{u} = 5\vec{i} + 12\vec{j}$

a) $\vec{u} \cdot \vec{v} = (5)(\frac{3}{5}) + (12)(\frac{4}{5}) = \frac{15}{5} + \frac{48}{5} = \frac{63}{5}$

$|\vec{v}| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$

$|\vec{u}| = \sqrt{(5)^2 + (12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

b) $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{(\frac{63}{5})}{(13)(1)} = \frac{63}{5(13)} = \frac{63}{65}$

c) $|\vec{u}|\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{(\frac{63}{5})}{(1)} = \frac{63}{5}$

d) $proj_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\frac{(\frac{63}{5})}{(1)^2}\right)\left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right) = \frac{63}{5}\left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right)$

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4) $\vec{v} = 2\vec{i} + 10\vec{j} - 11\vec{k}$, $\vec{u} = 2\vec{i} + 2\vec{j} + \vec{k}$

a) $\vec{u} \cdot \vec{v} = (2)(2) + (2)(10) + (1)(-11) = 4 + 20 - 11 = 13$

$|\vec{v}| = \sqrt{(2)^2 + (10)^2 + (-11)^2} = \sqrt{4 + 100 + 121} = \sqrt{225} = 15$

$|\vec{u}| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$

b) $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{(13)}{(3)(15)} = \frac{13}{45}$

c) $|\vec{u}|\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{(13)}{(15)} = \frac{13}{15}$

d) $proj_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\frac{(13)}{(15)^2}\right)\left(2\vec{i} + 10\vec{j} - 11\vec{k}\right)$

$= \frac{13}{225}\left(2\vec{i} + 10\vec{j} - 11\vec{k}\right)$

6) $\vec{v} = -\vec{i} + \vec{j}$, $\vec{u} = \sqrt{2}\,\vec{i} + \sqrt{3}\,\vec{j} + 2\,\vec{k}$

a) $\vec{u}\cdot\vec{v} = (\sqrt{2})(-1) + (\sqrt{3})(1) + (2)(0) = -\sqrt{2} + \sqrt{3} = \sqrt{3} - \sqrt{2}$

$|\vec{v}| = \sqrt{(-1)^2 + (1)^2 + (0)^2} = \sqrt{1+1} = \sqrt{2}$

$|\vec{u}| = \sqrt{(\sqrt{2})^2 + (\sqrt{3})^2 + (2)^2} = \sqrt{2+3+4} = \sqrt{9} = 3$

b) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{(\sqrt{3}-\sqrt{2})}{(3)(\sqrt{2})} = \dfrac{\sqrt{3}}{3\sqrt{2}} - \dfrac{\sqrt{2}}{3(\sqrt{2})} = \dfrac{1}{(\sqrt{3})(\sqrt{2})} - \dfrac{1}{3} = \dfrac{1}{\sqrt{6}} - \dfrac{1}{3}$

c) $|\vec{u}|\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \dfrac{(\sqrt{3}-\sqrt{2})}{(\sqrt{2})} = \dfrac{\sqrt{3}}{\sqrt{2}} - \dfrac{\sqrt{2}}{\sqrt{2}} = \sqrt{\dfrac{3}{2}} - 1$

d) $\text{proj}_{\vec{v}}\,\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\dfrac{\sqrt{3}-\sqrt{2}}{(\sqrt{2})^2}\right)(-\vec{i}+\vec{j}) = \dfrac{\sqrt{3}-\sqrt{2}}{2}(-\vec{i}+\vec{j})$

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8) $\vec{v} = \left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{3}} \right\rangle$, $\vec{u} = \left\langle \dfrac{1}{\sqrt{2}}, \dfrac{-1}{\sqrt{3}} \right\rangle$

a) $\vec{u}\cdot\vec{v} = \left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{2}}\right) + \left(\dfrac{-1}{\sqrt{3}}\right)\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{3}{6} - \dfrac{2}{6} = \dfrac{1}{6}$

$|\vec{v}| = \sqrt{\left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{1}{\sqrt{3}}\right)^2} = \sqrt{\dfrac{1}{2} + \dfrac{1}{3}} = \sqrt{\dfrac{5}{6}} = \sqrt{\dfrac{5(6)}{6(6)}} = \dfrac{\sqrt{30}}{6}$

$|\vec{u}| = \sqrt{\left(\dfrac{1}{\sqrt{2}}\right)^2 + \left(\dfrac{-1}{\sqrt{3}}\right)^2} = \sqrt{\dfrac{1}{2} + \dfrac{1}{3}} = \sqrt{\dfrac{5}{6}} = \sqrt{\dfrac{5(6)}{6(6)}} = \dfrac{\sqrt{30}}{6}$

b) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{\left(\frac{1}{6}\right)}{\left(\frac{\sqrt{30}}{6}\right)\left(\frac{\sqrt{30}}{6}\right)} = \dfrac{6}{30} = \dfrac{2}{10} = \dfrac{1}{5}$

c) $|\vec{u}|\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \dfrac{\left(\frac{1}{6}\right)}{\left(\frac{\sqrt{30}}{6}\right)} = \dfrac{1}{\sqrt{30}}$

d) $\text{proj}_{\vec{v}}\,\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\dfrac{\left(\frac{1}{6}\right)}{\left(\frac{\sqrt{30}}{6}\right)^2}\right)\left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{3}} \right\rangle = \left(\dfrac{6}{30}\right)\left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{3}} \right\rangle$

$= \dfrac{1}{5}\left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{3}} \right\rangle$

14) $A=(1,0)$, $B=(0,3)$, $C=(3,4)$, $D=(4,1)$

$\overrightarrow{AC} = \langle 3-1, 4-0 \rangle = \langle 2,4 \rangle$, $\overrightarrow{BD} = \langle 4-0, 1-3 \rangle = \langle 4,-2 \rangle$

$\overrightarrow{AC} \cdot \overrightarrow{BD} = (2)(4)+(4)(-2) = 8-8 = 0$

$\cos\theta = \dfrac{\overrightarrow{AC} \cdot \overrightarrow{BD}}{|\overrightarrow{AC}||\overrightarrow{BD}|} = \dfrac{(0)}{|\overrightarrow{AC}||\overrightarrow{BD}|} = 0$, so the angle measures $\frac{\pi}{2}$

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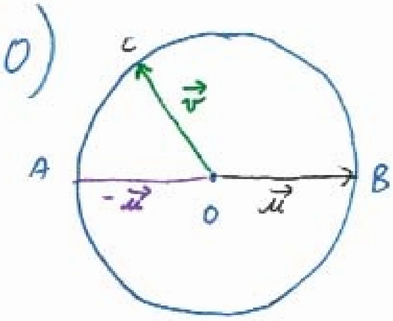
20)



$\overrightarrow{CA} = (-\vec{v}) + (-\vec{u}) = -\vec{v} - \vec{u}$

$\overrightarrow{CB} = (-\vec{v}) + (\vec{u}) = -\vec{v} + \vec{u}$

$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-\vec{v} - \vec{u}) \cdot (-\vec{v} + \vec{u})$

$= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{v} - \vec{u}\cdot\vec{u}$

$= \vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{u} = |\vec{v}|^2 - |\vec{u}|^2 = 0$

because $|\vec{u}| = |\vec{v}|$, both are the radius of this circle.

Therefore $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

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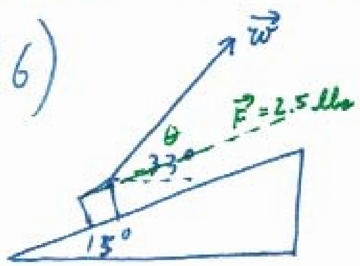
26)



$\theta = 33° - 15° = 18°$ $\quad \vec{F} = |\vec{w}| \cos 18°$

$|\vec{w}| = \dfrac{\vec{F}}{\cos 18°} = \dfrac{2.5 \text{ lbs}}{\cos 18°}$

direction of $\vec{w}$ is $\langle \cos 33°, \sin 33° \rangle$

so $\vec{w} = \dfrac{2.5 \text{ lbs}}{\cos 18°} \langle \cos 33°, \sin 33° \rangle$

30) No, $\vec{v}_1$ need not equal $\vec{v}_2$

i.e. $\vec{i}+\vec{j} \neq \vec{i}+2\vec{j}$

but $\vec{i}\cdot(\vec{i}+\vec{j}) = \vec{i}\cdot\vec{i}+\vec{i}\cdot\vec{j} = 1+0 = 1$

and $\vec{i}\cdot(\vec{i}+2\vec{j}) = \vec{i}\cdot\vec{i}+2\vec{i}\cdot\vec{j} = 1+2(0) = 1$

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44) $|\vec{F}| = 602148\,N$ $\quad \vec{D} = \overrightarrow{(SF)(LA)} = |\overrightarrow{(SF)(LA)}|\cos\theta$, $\theta = 0$

$|\vec{D}| = 605\,km$ $\quad W = |\vec{F}||\vec{D}|\cos\theta$

$W = (602148\,N)(605\,km)(\cos 0) = (602148\,N)(605000\,m)(1)$

$= (602148)(605000)\,J.$

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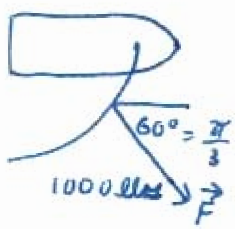
46) 

$|\vec{F}| = 1000\,lbs$, $|\vec{D}| = 1\,miles = 5280\,ft$

$\theta = \frac{\pi}{3}$ $\quad W = |\vec{F}||\vec{D}|\cos\theta$

$W = (1000\,lbs)(5280\,ft)\cos\frac{\pi}{3} = (1000\,lbs)(5280\,ft)(\tfrac{1}{2})$

$= (1000)(2640)\,ft\text{-}lbs = 2640000\,ft\text{-}lbs$