Component Form

Definitions

The vector represented by the directed line segment $\overline{AB}$ has **initial point** A and **terminal point** B and its **length** is denoted by $\left|\overline{AB}\right|$. Two vectors are **equal** if they have the same length and direction.

Definition

If $\mathbf{v} = \vec{v}$ is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the **component form** of $\mathbf{v} = \vec{v}$ is

$$\mathbf{v} = \vec{v} = \langle v_1, v_2 \rangle$$

If $\mathbf{v} = \vec{v}$ is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of $\mathbf{v} = \vec{v}$ is

$$\mathbf{v} = \vec{v} = \langle v_1, v_2, v_3 \rangle$$

The **magnitude** or **length** of the vector $\mathbf{v} = \vec{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = |\vec{v}| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Vector Algebra Operations

Definitions

Let $\mathbf{u} = \vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = k\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$

Properties of Algebra Operations

Let $\mathbf{u} = \vec{u}$, $\mathbf{v} = \vec{v}$, $\mathbf{w} = \vec{w}$ be vectors and a, b be scalars.

1.	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	$\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2.	$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3.	$\mathbf{u} + \mathbf{0} = \mathbf{u}$	$\vec{u} + \vec{0} = \vec{u}$
4.	$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$	$\vec{u} + (-\vec{u}) = \vec{0}$
5.	$0\mathbf{u} = \mathbf{0}$	$0\vec{u} = \vec{0}$
6.	$1\mathbf{u} = \mathbf{u}$	$1\vec{u} = \vec{u}$
7.	$a(b\mathbf{u}) = (ab)\mathbf{u}$	$a(b\vec{u}) = (ab)\vec{u}$
8.	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$	$a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
9.	$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$	$(a + b)\vec{u} = a\vec{u} + b\vec{u}$

Unit Vectors

A vector $\mathbf{v} = \vec{v}$ of length 1 is called a **unit vector**.

The **standard unit vectors** are $\mathbf{i} = \vec{i} = \langle 1, 0, 0 \rangle$ $\mathbf{j} = \vec{j} = \langle 0, 1, 0 \rangle$ $\mathbf{k} = \vec{k} = \langle 0, 0, 1 \rangle$

If $\mathbf{v} = \vec{v} \neq \mathbf{0} = \vec{0}$, then its length $|\mathbf{v}| = |\vec{v}|$ is not zero and

$$\left|\frac{1}{|\mathbf{v}|}\mathbf{v}\right| = \frac{1}{|\mathbf{v}|}|\mathbf{v}| = 1 \quad \text{or} \quad \left|\frac{1}{|\vec{v}|}\vec{v}\right| = \frac{1}{|\vec{v}|}|\vec{v}| = 1.$$

That is, $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{\vec{v}}{|\vec{v}|}$ is a unit vector in the direction of $\mathbf{v} = \vec{v}$, called **the direction** of the nonzero vector $\mathbf{v} = \vec{v}$.

If $\mathbf{v} = \vec{v} \neq \mathbf{0} = \vec{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{\vec{v}}{|\vec{v}|}$ is a unit vector called the direction of $\mathbf{v} = \vec{v}$
2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ or $\vec{v} = |\vec{v}|\dfrac{\vec{v}}{|\vec{v}|}$ express $\mathbf{v} = \vec{v}$ as its length times its direction.

Midpoint of a Line Segment

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

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$\vec{u} = \langle 3, -2 \rangle \qquad \vec{v} = \langle -2, 5 \rangle$

2) a) $-2\vec{v} = \langle -2(-2), -2(5) \rangle = \langle 4, -10 \rangle$

b) $|-2\vec{v}| = \sqrt{(4)^2 + (-10)^2} = \sqrt{16+100} = \sqrt{116} = 2\sqrt{29}$

4-a) $\vec{u} - \vec{v} = \langle 3-(-2), -2-5 \rangle = \langle 5, -7 \rangle$

b) $|\vec{u} - \vec{v}| = \sqrt{(5)^2 + (-7)^2} = \sqrt{25+49} = \sqrt{74}$

6-a) $-2\vec{u} = \langle -2(3), -2(-2) \rangle = \langle -6, 4 \rangle \qquad 5\vec{v} = \langle 5(-2), 5(5) \rangle = \langle -10, 25 \rangle$

$-2\vec{u} + 5\vec{v} = \langle -6 + (-10), 4 + 25 \rangle = \langle -16, 29 \rangle$

b) $|-2\vec{u} + 5\vec{v}| = \sqrt{(-16)^2 + (29)^2} = \sqrt{256 + 841} = \sqrt{1097}$

8-a) $\frac{-5}{13}\vec{u} = \langle \frac{-5}{13}(3), \frac{-5}{13}(-2) \rangle = \langle \frac{-15}{13}, \frac{10}{13} \rangle$

$\frac{12}{13}\vec{v} = \langle \frac{12}{13}(-2), \frac{12}{13}(5) \rangle = \langle \frac{-24}{13}, \frac{60}{13} \rangle$

$\frac{-5}{13}\vec{u} + \frac{12}{13}\vec{v} = \langle \frac{-15}{13} + \left(\frac{-24}{13}\right), \frac{10}{13} + \frac{60}{13} \rangle = \langle \frac{-39}{13}, \frac{70}{13} \rangle = \langle -3, \frac{70}{13} \rangle$

b) $\left| \frac{-5}{13}\vec{u} + \frac{12}{13}\vec{v} \right| = \sqrt{(-3)^2 + \left(\frac{70}{13}\right)^2} = \sqrt{\left(\frac{-39}{13}\right)^2 + \left(\frac{70}{13}\right)^2} = \sqrt{\frac{(-39)^2 + (70)^2}{(13)^2}}$

$= \frac{\sqrt{1521 + 4900}}{13} = \frac{\sqrt{6421}}{13}$

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10) $\vec{O}: \langle 0, 0 \rangle$; $R = (2, -1)$, $S = (-4, 3)$ $\qquad \vec{P} = \langle \frac{2+(-4)}{2}, \frac{-1+3}{2} \rangle = \langle -1, 1 \rangle$

$\overrightarrow{OP} = \langle -1-0, 1-0 \rangle = \langle -1, 1 \rangle$

14) $\theta = \frac{-3\pi}{4}$ $\quad x = 1\cos\left(\frac{-3\pi}{4}\right) = 1\left(\frac{-1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}}$ $\quad y = 1\sin\left(\frac{-3\pi}{4}\right) = 1\left(\frac{-1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}}$

vector: $\langle x, y\rangle = \left\langle \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$

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16) unit vector $\langle 1, 0\rangle$ $\;135°$ counterclockwise: $\theta = \frac{3\pi}{4}$

$x = 1\cos\left(\frac{3\pi}{4}\right) = \frac{-1}{\sqrt{2}}$ $\qquad y = 1\sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$

vector: $\langle x, y\rangle = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

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18) $P_1\ (1,2,0)$, $P_2\ (-3,0,5)$

$\overrightarrow{P_1P_2} = (-3-1)\vec{i} + (0-2)\vec{j} + (5-0)\vec{k} = -4\vec{i} - 2\vec{j} + 5\vec{k}$

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20) $A\ (1,0,3)$, $B\ (-1,4,5)$

$\overrightarrow{AB} = (-1-1)\vec{i} + (4-0)\vec{j} + (5-3)\vec{k} = -2\vec{i} + 4\vec{j} + 2\vec{k}$

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22) $\vec{u} = \langle -1, 0, 2\rangle$, $\vec{v} = \langle 1, 1, 1\rangle$

$-2\vec{u} + 3\vec{v} = -2\langle -1,0,2\rangle + 3\langle 1,1,1\rangle = \langle 2,0,-4\rangle + \langle 3,3,3\rangle$

$= \langle 2+3, 0+3, -4+3\rangle = \langle 5, 3, -1\rangle = 5\vec{i} + 3\vec{j} - \vec{k}$

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26) $9\vec{i} - 2\vec{j} + 6\vec{k}$

length: $|9\vec{i} - 2\vec{j} + 6\vec{k}| = \sqrt{(9)^2 + (-2)^2 + (6)^2} = \sqrt{81+4+36} = \sqrt{121} = 11$

direction: $\frac{9\vec{i} - 2\vec{j} + 6\vec{k}}{11} = \frac{9}{11}\vec{i} - \frac{2}{11}\vec{j} + \frac{6}{11}k$

$9\vec{i} - 2\vec{j} + 6\vec{k} = 11\left(\frac{9}{11}\vec{i} - \frac{2}{11}\vec{j} + \frac{6}{11}\vec{k}\right)$

28) $\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$

length: $\left|\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$

direction: $\frac{\frac{3}{5}}{1}\vec{i} + \frac{\frac{4}{5}}{1}\vec{j} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$

$\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j} = 1\left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right)$

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30) $\frac{\vec{i}}{\sqrt{3}} + \frac{\vec{j}}{\sqrt{3}} + \frac{\vec{k}}{\sqrt{3}}$

length: $\left|\frac{\vec{i}}{\sqrt{3}} + \frac{\vec{j}}{\sqrt{3}} + \frac{\vec{k}}{\sqrt{3}}\right| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \sqrt{\frac{3}{3}} = 1$

direction: $\frac{\frac{1}{\sqrt{3}}}{1}\vec{i} + \frac{\frac{1}{\sqrt{3}}}{1}\vec{j} + \frac{\frac{1}{\sqrt{3}}}{1}\vec{k} = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}$

$\frac{\vec{i}}{\sqrt{3}} + \frac{\vec{j}}{\sqrt{3}} + \frac{\vec{k}}{\sqrt{3}} = 1\left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right)$

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32) a) $7(-\vec{j}) = -7\vec{j}$

b) $\sqrt{2}\left(\frac{-3}{5}\vec{i} - \frac{4}{5}\vec{k}\right) = \frac{-3\sqrt{2}}{5}\vec{i} - \frac{4\sqrt{2}}{5}\vec{k}$

c) $\frac{13}{12}\left(\frac{3}{13}\vec{i} - \frac{4}{13}\vec{j} - \frac{12}{13}\vec{k}\right) = \frac{1}{4}\vec{i} - \frac{1}{3}\vec{j} - \vec{k}$

d) $a\left(\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{6}}\vec{k}\right) = \frac{a}{\sqrt{2}}\vec{i} + \frac{a}{\sqrt{3}}\vec{j} - \frac{a}{\sqrt{6}}\vec{k}$, $a > 0$

36) $P_1(1,4,5)$, $P_2(4,-2,7)$

a) $\overrightarrow{P_1P_2} = (4-1)\vec{i} + (-2-4)\vec{j} + (7-5)\vec{k} = 3\vec{i} - 6\vec{j} + 2\vec{k}$

$|\overrightarrow{P_1P_2}| = \sqrt{(3)^2+(-6)^2+(2)^2} = \sqrt{9+36+4} = \sqrt{49} = 7$

direction is $\frac{3}{7}\vec{i} - \frac{6}{7}\vec{j} + \frac{2}{7}\vec{k}$

b) midpoint is $\left(\frac{1+4}{2}, \frac{4+(-2)}{2}, \frac{5+7}{2}\right) = \left(\frac{5}{2}, 1, 6\right)$

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38) $P_1(0,0,0)$, $P_2(2,-2,-2)$

a) $\overrightarrow{P_1P_2} = (2-0)\vec{i} + (-2-0)\vec{j} + (-2-0)\vec{k} = 2\vec{i} - 2\vec{j} - 2\vec{k}$

$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2+(-2)^2+(-2)^2} = \sqrt{4+4+4} = \sqrt{4(1+1+1)} = \sqrt{4(3)} = 2\sqrt{3}$

direction is $\frac{2}{2\sqrt{3}}\vec{i} - \frac{2}{2\sqrt{3}}\vec{j} - \frac{2}{2\sqrt{3}}\vec{k} = \frac{1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}$

b) midpoint is $\left(\frac{2+0}{2}, \frac{-2+0}{2}, \frac{-2+0}{2}\right) = (1,-1,-1)$

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42) $\vec{u} = \vec{i} - 2\vec{j}$, $\vec{v} = 2\vec{i} + 3\vec{j}$, $\vec{w} = \vec{i} + \vec{j}$

$\vec{u} = \vec{u}_1 + \vec{u}_2$ where $\vec{u}_1$ is $\parallel$ to $\vec{v}$ and $\vec{u}_2$ is $\parallel$ to $\vec{w}$

since $\vec{u}_1$ is $\parallel$ to $\vec{v}$: $\vec{u}_1 = a(2\vec{i} + 3\vec{j}) = 2a\vec{i} + 3a\vec{j}$

$\vec{u}_2$ is $\parallel$ to $\vec{w}$: $\vec{u}_2 = b(\vec{i} + \vec{j}) = b\vec{i} + b\vec{j}$

$\vec{i} - 2\vec{j} = \vec{u} = \vec{u}_1 + \vec{u}_2 = 2a\vec{i} + 3a\vec{j} + b\vec{i} + b\vec{j}$

so $\vec{i} = 2a\vec{i} + b\vec{i} \Rightarrow 1 = 2a + b \Rightarrow b = 1-2a$

$-2\vec{j} = 3a\vec{j} + b\vec{j} \Rightarrow -2 = 3a + b$

$$\left.\begin{aligned} -2 &= 3a + (1-2a) \\ -2 &= a + 1 \\ -3 &= a \end{aligned}\right. \qquad b = 1 - 2(-3) = 7$$

$\vec{u}_1 = 2(-3)\vec{i} + 3(-3)\vec{j} = -6\vec{i} - 9\vec{j}$ $\quad$ $\vec{u}_2 = (7)\vec{i} + (7)\vec{j} = 7\vec{i} + 7\vec{j}$