Definition

If x and y are given as functions

$$x = f(t) \qquad y = g(t),$$

over an interval *I* of *t*-values, then the set of points $(x, y) = (f(t)), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

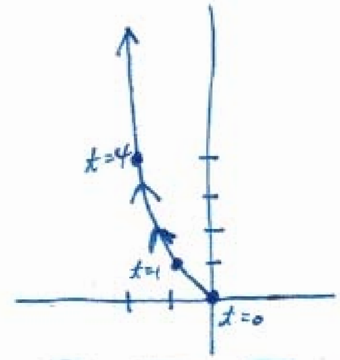
2) $x = -\sqrt{t}$, $y = t$, $t \geq 0$

t	x	y
0	0	0
1	-1	1
4	-2	4

$x = -\sqrt{y}$

or

$y = x^2,\ x \leq 0$



4) $x = 3-3t$, $y = 2t$, $0 \leq t \leq 1$

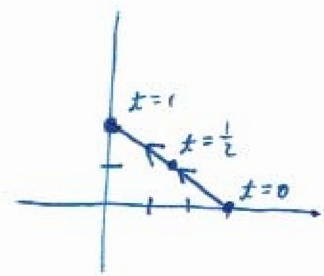
t	x	y
0	3	0
$\frac{1}{2}$	$\frac{3}{2}$	1
1	0	2

$x = 3-3t$

$3t = 3-x$

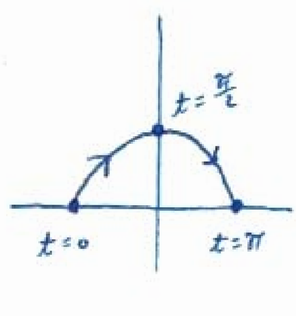
$t = 1-\frac{x}{3}$

$y = 2(1-\frac{x}{3}) = 2-\frac{2}{3}x,\ 0 \leq x \leq 3$



6) $x = \cos(\pi - t)$, $y = \sin(\pi - t)$, $0 \leq t \leq \pi$

t	x	y
0	-1	0
$\frac{\pi}{2}$	0	1
π	1	0



$(\cos(\pi-t))^2 + (\sin(\pi-t))^2 = 1^2$

$\cos^2(\pi-t) + \sin^2(\pi-t) = 1$

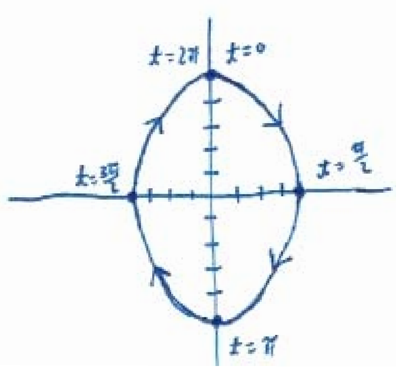
$x^2 + y^2 = 1,\ y \geq 0$

$y^2 = 1-x^2$

$y = \pm\sqrt{1-x^2} \Rightarrow y = +\sqrt{1-x^2}$

8) $x = 4\sin t$, $y = 5\cos t$, $0 \leq t \leq 2\pi$

t	x	y
0	0	5
$\frac{\pi}{2}$	4	0
π	0	-5
$\frac{3\pi}{2}$	-4	0
2π	0	5



$x = 4\sin t \quad y = 5\cos t$

$\frac{x}{4} = \sin t \quad \frac{y}{5} = \cos t$

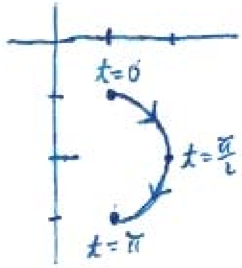
$\sin^2 t + \cos^2 t = 1$

$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$

$\frac{x^2}{16} + \frac{y^2}{25} = 1$

10) $x = 1 + \sin t$, $y = \cos t - 2$, $0 \le t \le \pi$

t	x	y
0	1	-1
$\frac{\pi}{2}$	2	-2
π	1	-3



$x = 1 + \sin t \qquad y = \cos t - 2$

$x - 1 = \sin t \qquad y + 2 = \cos t$

$\sin^2 t + \cos^2 t = 1$

$(x-1)^2 + (y+2)^2 = 1$, $1 \le x \le 2$

$(y+2)^2 = 1 - (x-1)^2$

$y + 2 = \pm\sqrt{1-(x-1)^2}$

$y = -2 \pm \sqrt{1-(x-1)^2} \Rightarrow y = -2 + \sqrt{1-(x-1)^2}$

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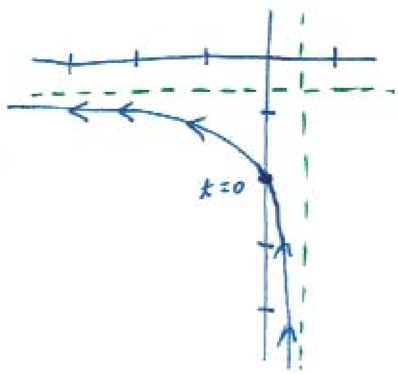
12) $x = \frac{t}{t-1}$, $y = \frac{t-2}{t+1}$, $-1 < t < 1$

t	x	y
-1	$\frac{1}{2}$	undefined $-\infty$
0	0	-2
1	undefined $+\infty$	$\frac{-1}{2}$



$x = \frac{t}{t-1}$

$x(t-1) = t$

$xt - x = t$

$xt - t = x$

$t(x-1) = x$

$t = \frac{x}{x-1}$

$$y = \left(\frac{\left(\frac{x}{x-1}\right) - 2}{\left(\frac{x}{x-1}\right) + 1}\right)\left(\frac{\frac{x-1}{1}}{\frac{x-1}{1}}\right) = \frac{x - 2(x-1)}{x + 1(x-1)}$$

$$y = \frac{x - 2x + 2}{x + x - 1} = \frac{-x+2}{2x+1} = \frac{2-x}{2x+1}$$

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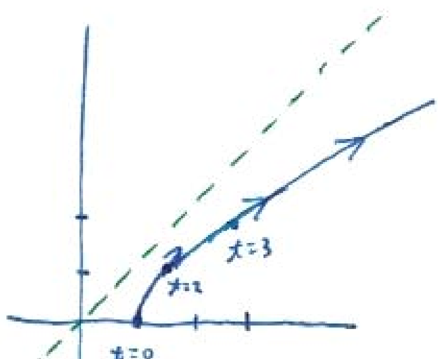
14) $x = \sqrt{t+1}$, $y = \sqrt{t}$, $t \ge 0$

t	x	y
0	1	0
2	$\sqrt{3}$	$\sqrt{2}$
3	2	$\sqrt{3}$
8	3	$\sqrt{8} = 2\sqrt{2}$



$y = \sqrt{t}$

$y^2 = t$

$x = \sqrt{y^2+1}$, $y \ge 0$

or

$x = \sqrt{t+1}$

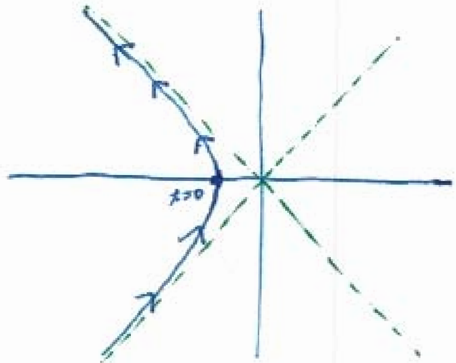
$x^2 = t + 1$

$x^2 - 1 = t$

$y = \sqrt{x^2-1}$, $x \ge 1$

16) $x = -\sec t$, $y = \tan t$, $\frac{-\pi}{2} < t < \frac{\pi}{2}$

t	x	y
$\frac{-\pi}{2}$	undefined $-\infty$	undefined $-\infty$
0	-1	0
$\frac{\pi}{2}$	undefined $-\infty$	undefined $+\infty$



$(-\sec t)^2 - (\tan t)^2 = 1$

$\sec^2 t - \tan^2 t = 1$

$x^2 - y^2 = 1$, $x < 0$

or

$x^2 = y^2 + 1$

$x = \pm\sqrt{y^2+1}$

$x = -\sqrt{y^2+1}$

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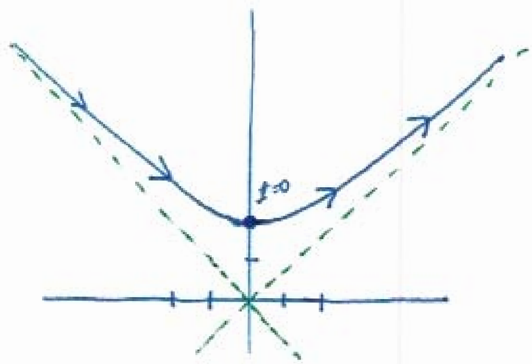
18) $x = 2\sinh t$, $y = 2\cosh t$, $-\infty < t < \infty$

t	x	y
-1	$2\left(\frac{e^{-1} - e^{1}}{2}\right)$	$2\left(\frac{e^{-1} + e^{1}}{2}\right)$
0	0	$2\left(\frac{e^{0} + e^{-0}}{2}\right) = 2$
1	$2\left(\frac{e^{1} - e^{-1}}{2}\right)$	$2\left(\frac{e^{1} + e^{-1}}{2}\right)$



$\cosh^2 t - \sinh^2 t = 1$

$x = 2\sinh t$ $\quad y = 2\cosh t$

$\frac{x}{2} = \sinh t$ $\quad \frac{y}{2} = \cosh t$

$\left(\frac{y}{2}\right)^2 - \left(\frac{x}{2}\right)^2 = 1$

$\frac{y^2}{4} - \frac{x^2}{4} = 1$

$y^2 - x^2 = 4$, $y > 0$

$y^2 = x^2 + 4$

$y = \pm\sqrt{x^2+4} \Rightarrow y = +\sqrt{x^2+4}$

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20) $x = \cos t$, $y = 2\sin t$

both x, y have period of 2π by the amplitude of x is 1 and of y is 2; so it will produce a graph of an ellipse. Picture B

22) $x = \sqrt{t}$, $y = \sqrt{t}\cos t$

both x, y are defined for $[0, \infty)$; while x will be all positive, y will be going from positive, to negative, to positive, to negative thus oscillating while increasing by factor of $\sqrt{t}$. Picture A.

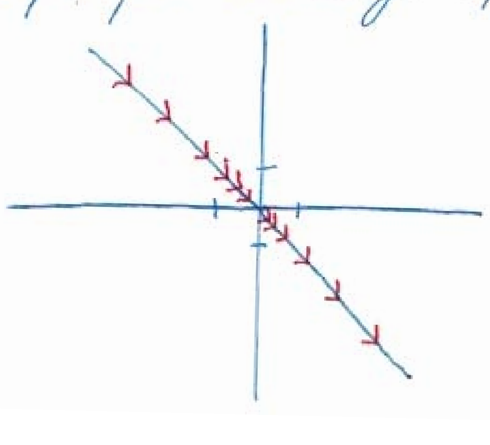
24) $x = \cos t$, $y = \sin 3t$

x has period of 2π and y has period of $\frac{2\pi}{3}$.
This means that y will oscillate 3 times quicker than x. Picture F

26) $x = f(t) = +t^3$ $\quad$ $y = g(t) = -t^3$

x increases and y decreases for $-\infty < t < \infty$

also note the amount of change is small near $t = 0$ for both x and y while the quantity is identical except for the sign (positive & negative)



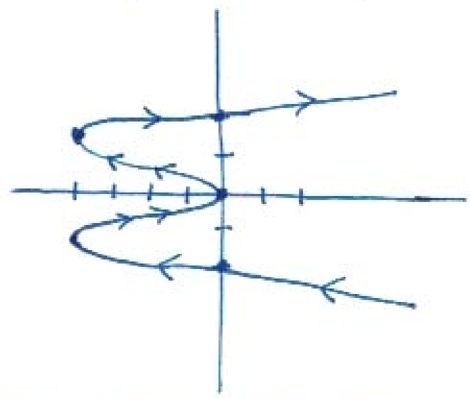
the arrow head marked in red shows the pace of the movement

28) $x = f(t)$ a 4th degree polynomial opening upward so the value of x; decreases to -4, increases to 0, decreases to -4, increases

$y = g(t) = t$ which is an increasing linear function

here are some values as table

t	x	y
-2	0	-2
0	0	0
2	0	2



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30) starts $(a, 0)$ ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ $(\cos t)^2 + (\sin t)^2 = 1$

$\frac{x}{a} = \cos t$ $\frac{y}{b} = \sin t$

$x = a\cos t$ $y = b\sin t$

a) once clockwise:

$x = a\cos t$, $y = -b\sin t$

$0 \le t \le 2\pi$

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b) once counter clockwise:

$x = a\cos t$, $y = b\sin t$

$0 \le t \le 2\pi$

---

c) twice clockwise: $x = a\cos t$, $y = -b\sin t$, $0 \le t \le 4\pi$

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d) twice counter clockwise: $x = a\cos t$, $y = b\sin t$, $0 \le t \le 4\pi$

32) line segment with endpoints $(-1,3)$ and $(3,-2)$

start with point $(-1,3)$ and create the parametric equations $x=-1+at$ and $y=3+bt$. This equation will have the point $(-1,3)$ when $t=0$.

Now we need to find a and b so it ends at point $(3,-2)$ and to make our calculation simpler, we pick $t=1$.

$x=-1+at$	$y=3+bt$	so $x=-1+4t$
$(3)=-1+a(1)$	$(-2)=3+b(1)$	
$3=-1+a$	$-2=3+b$	$y=3-5t$
$4=a$	$-5=b$	$0\le t\le 1$

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34) the left half of the parabola $y=x^2+2x$

$$y=x^2+2x=x^2+2x+1-1=(x+1)^2-1$$

so the vertex is $(-1,-1)$ and $x\le -1$

since this is a parabola opening upward, the value of x can increase linearly and y must be quadratic so $x=t$ and using substitution we get

$y=(t+1)^2-1=t^2+2t$ so $x=t$, $y=t^2+2t$, $t\le -1$

38) $y = x^2$ $\quad (0,0) \to (3,9) \to (0,0) \to (-3,9) \to (0,0) \to (3,9) \to \dots$

we need to make our path oscillate, so we must use $\sin t$ or $\cos t$. Since we are starting from $(0,0)$ and x will vary $[-3,3]$ while y will vary $[0,9]$

let $x = 3\sin t$ and $y = (3\sin t)^2 = 9\sin^2 t$, $0 \le t < \infty$

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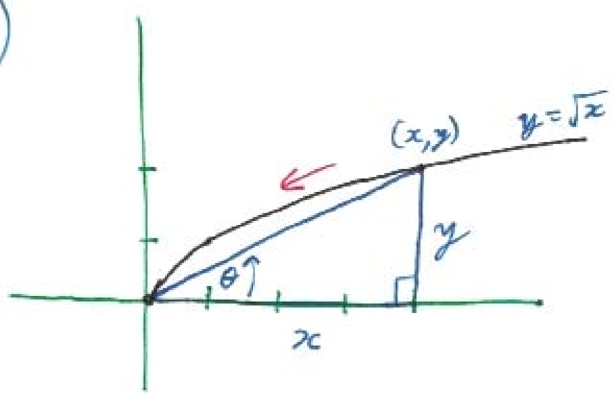
42)



$\tan\theta = \dfrac{y}{x}$

$x\tan\theta = y$

$y = \sqrt{x}$

$\Downarrow$

$y^2 = x$

$(x\tan\theta)^2 = x$

$x^2\tan^2\theta = x$

$\dfrac{x^2}{x} = \dfrac{1}{\tan^2\theta}$

$x = \cot^2\theta$

$y = \sqrt{x}$

$y = \sqrt{\cot^2\theta}$

$y = \cot\theta$

so $\boxed{\begin{array}{l} x = \cot^2\theta \\ y = \cot\theta \\ 0 < \theta \le \frac{\pi}{2} \end{array}}$