The Taylor series generated by $f(x) = (1+x)^m$, where m is constant, is

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}x^k + \cdots. \qquad (1)$$

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2(n+1)}}{1+t^2} \qquad (2)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2x+1}}{2n+1} \qquad |x| \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad |x| \leq 1 \qquad (3)$$

Definition For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$. (4)

Table 10.1 Frequently Used Taylor Series (from section 10.10)

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad |x| \leq 1$$

$$2)\ (1+x)^{\frac{1}{3}} = 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{3}}{k} x^k = 1 + \left(\tfrac{1}{3}\right)x^1 + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}x^2 + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^3 + \cdots$$

$$= 1 + \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)}{2}x^2 + \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{(2)(3)}x^3 + \cdots$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \cdots$$

$$4)\ (1-2x)^{\frac{1}{2}} = (1+(-2x))^{\frac{1}{2}} = 1 + \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (-2x)^k$$

$$= 1 + \left(\tfrac{1}{2}\right)(-2x)^1 + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-2x)^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-2x)^3 + \cdots$$

$$= 1 - x + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2}(4x^2) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{(2)(3)}(-8x^3) + \cdots$$

$$= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \cdots$$

$$6)\ \left(1-\frac{x}{3}\right)^4 = \left(1+\left(\frac{-x}{3}\right)\right)^4 = 1 + \sum_{k=1}^{\infty} \binom{4}{k} \left(\frac{-x}{3}\right)^k$$

$$= 1 + (4)\left(\frac{-x}{3}\right)^1 + \frac{(4)(4-1)}{2!}\left(\frac{-x}{3}\right)^2 + \frac{(4)(4-1)(4-2)}{3!}\left(\frac{-x}{3}\right)^3 + \cdots$$

$$= 1 - \frac{4}{3}x + \frac{(4)(3)}{2}\left(\frac{x^2}{9}\right) + \frac{(4)(3)(2)}{(2)(3)}\left(\frac{-x^3}{27}\right) + \cdots$$

$$= 1 - \frac{4}{3}x + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \cdots$$

$$= 1 - \frac{4}{3}x + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{(4)(3)(2)(1)}{(2)(3)(4)}\left(\frac{-x}{3}\right)^4 + \frac{(4)(3)(2)(1)(0)}{5!}\left(\frac{-x}{3}\right)^5 + \cdots$$

$$= 1 - \frac{4}{3}x + \frac{2}{3}x^2 - \frac{4}{27}x^3 + \frac{1}{81}x^4 + 0 + 0 + \cdots$$

8) $(1+x^2)^{\frac{-1}{3}} = 1 + \sum_{k=1}^{\infty} \binom{\left(\frac{-1}{3}\right)}{k} (x^2)^k = 1 + \left(\frac{-1}{3}\right)(x^2)^1 + \frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)}{2!}(x^2)^2 + \frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{3!}(x^2)^3 + \cdots$

$= 1 - \frac{1}{3}x^2 + \frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)}{2}(x^4) + \frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\left(\frac{-7}{3}\right)}{(2)(3)}(x^6) + \cdots$

$= 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 + \cdots$

---

10) $\frac{x}{\sqrt[3]{1+x}} = x(1+x)^{\frac{-1}{3}} = x\left\{1 + \sum_{k=1}^{\infty}\binom{\left(\frac{-1}{3}\right)}{k}x^k\right\}$

$= x\left\{1 + \left(\frac{-1}{3}\right)(x)^1 + \frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)}{2!}(x)^2 + \frac{\left(\frac{-1}{3}\right)\left(\frac{-1}{3}-1\right)\left(\frac{-1}{3}-2\right)}{3!}(x)^3 + \cdots\right\}$

$= x\left\{1 - \frac{1}{3}x + \frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)}{2}x^2 \cdot \frac{\left(\frac{-1}{3}\right)\left(\frac{-4}{3}\right)\left(\frac{-7}{3}\right)}{(2)(3)}x^3 + \cdots\right\}$

$= x - \frac{1}{3}x^2 + \frac{2}{9}x^3 - \frac{14}{81}x^4 + \cdots$

---

12) $(1+x)^4 = 1 + \sum_{k=1}^{\infty}\binom{4}{k}x^k = 1 + (4)(x)^1 + \frac{(4)(4-1)}{2!}(x)^2 + \frac{(4)(4-1)(4-2)}{3!}(x)^3 + \frac{(4)(4-1)(4-2)(4-3)}{4!}(x)^4$

$= 1 + 4x + \frac{(4)(3)}{2}x^2 + \frac{(4)(3)(2)}{(2)(3)}x^3 + \frac{(4)(3)(2)(1)}{(2)(3)(4)}x^4 + 0 \cdots$

$= 1 + 4x + 6x^2 + 4x^3 + x^4$

---

14) $\left(1 - \frac{x}{2}\right)^4 = \left(1 + \left(\frac{-x}{2}\right)\right)^4 = 1 + \sum_{k=1}^{\infty}\binom{4}{k}\left(\frac{-x}{2}\right)^k$

$= 1 + (4)\left(\frac{-x}{2}\right)^1 + \frac{(4)(4-1)}{2!}\left(\frac{-x}{2}\right)^2 + \frac{(4)(4-1)(4-2)}{3!}\left(\frac{-x}{2}\right)^3 + \frac{(4)(4-1)(4-2)(4-3)}{4!}\left(\frac{-x}{2}\right)^4$

$= 1 - 2x + \frac{(4)(3)}{(2)}\left(\frac{x^2}{4}\right) + \frac{(4)(3)(2)}{(2)(3)}\left(\frac{-x^3}{8}\right) + \frac{(4)(3)(2)(1)}{(2)(3)(4)}\left(\frac{x^4}{16}\right) + 0 \cdots$

$= 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

24) $\cos\sqrt{t}$ approximated by $1-\frac{t}{2!}+\frac{t^2}{4!}-\frac{t^3}{6!}$

the error will be $|R_5(x)|$ (5th term)

$$\int_0^1 \cos\sqrt{t}\,dt = \int_0^1\left(1-\frac{t}{2!}+\frac{t^2}{4!}-\frac{t^3}{6!}+\frac{t^4}{8!}-\cdots\right)dt$$

$$=\Bigg[t-\frac{t^2}{2(2!)}+\frac{t^3}{3(4!)}-\frac{t^4}{4(6!)}+\underbrace{\frac{t^5}{5(8!)}}_{}-\cdots\Bigg]_0^1$$

taking this term (5th)

$$|\text{error}| = |R_5(x)| \le \left|\frac{(1)^5}{5(8!)}\right| = \frac{1}{5(8!)} = \frac{1}{5(1)(2)(3)(4)(5)(6)(7)(8)}$$

---

26) $F(x)=\int_0^x t^2e^{-t^2}\,dt \quad [0,1] \quad \text{error} < 10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$

$$t^2e^{-t^2} = t^2\left(1+(-t^2)+\frac{(-t^2)^2}{2!}+\frac{(-t^2)^3}{3!}+\frac{(-t^2)^4}{4!}+\frac{(-t^2)^5}{5!}+\frac{(-t^2)^6}{6!}+\cdots\right)$$

$$= t^2-t^4+\frac{t^6}{2!}-\frac{t^8}{3!}+\frac{t^{10}}{4!}-\frac{t^{12}}{5!}+\frac{t^{14}}{6!}-\cdots$$

$$F(x)=\int_0^x t^2e^{-t^2}dt = \int_0^x\left(t^2-t^4+\frac{t^6}{2!}-\frac{t^8}{3!}+\frac{t^{10}}{4!}-\frac{t^{12}}{5!}+\frac{t^{14}}{6!}-\cdots\right)dt$$

$$=\left[\frac{1}{3}t^3-\frac{1}{5}t^5+\frac{1}{7(2!)}t^7-\frac{1}{9(3!)}t^9+\frac{1}{11(4!)}t^{11}-\frac{1}{13(5!)}t^{13}+\frac{1}{15(6!)}t^{15}+\cdots\right]_0^x$$

$$=\left[\frac{1}{3}x^3-\frac{1}{5}x^5+\frac{1}{7(2)}x^7-\frac{1}{9(3!)}x^9+\underbrace{\frac{1}{11(4!)}}_{}x^{11}-\underbrace{\frac{1}{13(5!)}}_{}x^{13}+\frac{1}{15(6!)}x^{15}+\cdots\right]-[0]$$

$$\frac{1}{(11)(4)(3)(2)(1)} = \frac{1}{264}(1)^{11} \qquad \frac{1}{(13)(5)(4)(3)(2)(1)} = \frac{1}{1560}(1)^{13}$$

26) continued

so we should set $F(x) = \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{14}x^7 - \frac{1}{9(3!)}x^9 + \frac{1}{11(4!)}x^{11}$

where the error is $|\text{error}| < \frac{1}{1560}$

---

28) $F(x) = \int_0^x \frac{\ln(1+t)}{t}\,dt$ error $< 10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$

$$\frac{\ln(1+t)}{t} = \frac{1}{t}\ln(1+t) = \frac{1}{t}\left\{t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \frac{t^6}{6} + \dots\right\}$$

$$= 1 - \tfrac{1}{2}t + \tfrac{1}{3}t^2 - \tfrac{1}{4}t^3 + \tfrac{1}{5}t^4 - \tfrac{1}{6}t^5 + \dots$$

$$F(x) = \int_0^x \frac{\ln(1+t)}{t}\,dt = \int_0^x \left(1 - \tfrac{1}{2}t + \tfrac{1}{3}t^2 - \tfrac{1}{4}t^3 + \tfrac{1}{5}t^4 - \tfrac{1}{6}t^5 + \dots\right)dt$$

$$= \left[t - \frac{1}{2(2)}t^2 + \frac{1}{3(3)}t^3 - \frac{1}{4(4)}t^4 + \frac{1}{5(5)}t^5 - \frac{1}{6(6)}t^6 + \dots\right]_0^x$$

$$= \left[x - \frac{1}{2^2}x^2 + \frac{1}{3^2}x^3 - \frac{1}{4^2}x^4 + \frac{1}{5^2}x^5 - \frac{1}{6^2}x^6 + \dots\right] - [0]$$

a) for this part let $x = \frac{1}{2}$ 5th term: $\frac{1}{5^2}\left(\frac{1}{2}\right)^5 = \frac{1}{800} > \frac{1}{1000}$

6th term: $\frac{1}{6^2}\left(\frac{1}{2}\right)^6 = \frac{1}{2304} < \frac{1}{1000}$

so $F(x) = x - \frac{1}{2^2}x^2 + \frac{1}{3^2}x^3 - \frac{1}{4^2}x^4 + \frac{1}{5^2}x^5$ with $|\text{error}| < \frac{1}{2304}$

b) for this part let $x = 1$ which means that we need $|\text{error}| < \frac{1}{k^2} \le \frac{1}{1000}$

31st term: $\frac{1}{31^2} = \frac{1}{961} > \frac{1}{1000}$ 32nd term: $\frac{1}{32^2} = \frac{1}{1024} < \frac{1}{1000}$

so $F(x) = x - \frac{1}{2^2}x^2 + \frac{1}{3^2}x^3 - \frac{1}{4^2}x^4 + \dots + (-1)^{30}\frac{x^{31}}{31^2} = \sum_{n=1}^{31}(-1)^{n-1}\frac{x^n}{n^2}$

with $|\text{error}| < \frac{1}{1024}$

30) $$\frac{e^x - e^{-x}}{x} = \frac{1}{x}\left\{e^x - e^{-x}\right\} = \frac{1}{x}\left\{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right) - \left(1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\cdots\right)\right\}$$

$$= \frac{1}{x}\left\{2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots\right\} = 2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \cdots$$

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{x} = \lim_{x\to 0}\left\{2 + \frac{2x^2}{3!} + \frac{2x^4}{5!} + \cdots\right\} = 2+0+0+\cdots = 2$$

---

32) $$\frac{\sin\theta - \theta + \frac{\theta^3}{6}}{\theta^5} = \frac{1}{\theta^5}\left\{-\theta + \frac{\theta^3}{6} + \sin\theta\right\}$$

$$= \frac{1}{\theta^5}\left\{-\theta + \frac{\theta^3}{6} + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \cdots\right)\right\}$$

$$= \frac{1}{\theta^5}\left\{\frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \frac{\theta^9}{9!} - \frac{\theta^{11}}{11!} + \cdots\right\}$$

$$= \left\{\frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \frac{\theta^6}{11!} + \cdots\right\}$$

$$\lim_{\theta\to 0} \frac{\sin\theta - \theta + \frac{\theta^3}{6}}{\theta^5} = \lim_{\theta\to 0}\left\{\frac{1}{5!} - \frac{\theta^2}{7!} + \frac{\theta^4}{9!} - \frac{\theta^6}{11!} + \cdots\right\}$$

$$= \frac{1}{5!} - 0 + 0 - 0 + \cdots = \frac{1}{5!} = \frac{1}{(5)(4)(3)(2)(1)} = \frac{1}{120}$$

---

34) $$\frac{\tan^{-1}y - \sin y}{y^3\cos y} = \frac{\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)}{y^3\cos y}$$

$$= \frac{\left(y - \frac{(2!)y^3}{3!} + \frac{(4!)y^5}{5!} - \cdots\right) - \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right)}{y^3\cos y} = \frac{\frac{1-2!}{3!}y^3 + \frac{4!-1}{5!}y^5 + \cdots}{y^3\cos y}$$

$$= \frac{\frac{1-2!}{3!} + \frac{4!-1}{5!}y^2 + \cdots}{\cos y}$$

34) continued

$$\lim_{y\to 0}\frac{\tan^{-1}y-\sin y}{y^3\cos y}=\lim_{y\to 0}\left\{\frac{\frac{1-2!}{3!}+\frac{4!-1}{5!}y^2+\cdots}{\cos y}\right\}=\frac{\frac{1-(2)}{(2)(3)}+\frac{(4)(3)(2)(1)-1}{(5)(4)(3)(2)(1)}(0)^2+\cdots}{\cos(0)}$$

$$=\frac{\frac{-1}{6}+\frac{23}{120}(0)^2+0+\cdots}{(1)}=\frac{-1}{6}$$

---

36) $$(x+1)\sin\left(\frac{1}{x+1}\right)=(x+1)\left\{\left(\frac{1}{x+1}\right)-\frac{\left(\frac{1}{x+1}\right)^3}{3!}+\frac{\left(\frac{1}{x+1}\right)^5}{5!}-\frac{\left(\frac{1}{x+1}\right)^7}{7!}+\cdots\right\}$$

$$=\left\{1-\frac{1}{3!(x+1)^2}+\frac{1}{5!(x+1)^4}-\frac{1}{7!(x+1)^6}+\cdots\right\}$$

$$\lim_{x\to\infty}(x+1)\sin\left(\frac{1}{x+1}\right)=\lim_{x\to\infty}\left\{1-\frac{1}{3!(x+1)^2}+\frac{1}{5!(x+1)^4}-\frac{1}{7!(x+1)^6}+\cdots\right\}$$

$$=1-0+0-0+\cdots=1$$

---

38) $$\frac{x^2-4}{\ln(x-1)}=\frac{(x+2)(x-2)}{\ln(1+(x-2))}=\frac{(x+2)(x-2)}{\left\{(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}-\frac{(x-2)^4}{4}+\cdots\right\}}$$

$$=\frac{(x+2)(x-2)}{(x-2)\left\{1-\frac{(x-2)}{2}+\frac{(x-2)^2}{3}-\frac{(x-2)^3}{4}+\cdots\right\}}=\frac{x+2}{1-\frac{(x-2)}{2}+\frac{(x-2)^2}{3}-\frac{(x-2)^3}{4}+\cdots}$$

$$\lim_{x\to 2}\frac{x^2-4}{\ln(x-1)}=\lim_{x\to 2}\left\{\frac{x+2}{1-\frac{(x-2)}{2}+\frac{(x-2)^2}{3}-\frac{(x-2)^3}{4}+\cdots}\right\}=\frac{(2)+2}{1-\frac{((2)-2)}{2}+\frac{((2)-2)^2}{3}-\frac{((2)-2)^3}{4}+\cdots}$$

$$=\frac{4}{1-0+0-0+\cdots}=\frac{4}{1}=4$$

40) $\ln(1+x^3) = (x^3)^1 - \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} - \frac{(x^3)^4}{4} + \cdots$

$= x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots$

$x \sin(x^2) = x\left\{(x^2)^1 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots\right\}$

$= x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \cdots$

$$\frac{\ln(1+x^3)}{x\sin(x^2)} = \frac{x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots}{x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \cdots} = \frac{x^3\left(1 - \frac{x^3}{2} + \frac{x^6}{3} - \frac{x^9}{4} + \cdots\right)}{x^3\left(1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \cdots\right)}$$

$$= \frac{1 - \frac{x^3}{2} + \frac{x^6}{3} - \frac{x^9}{4} + \cdots}{1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \cdots}$$

$$\lim_{x \to 0} \frac{\ln(1+x^2)}{x\sin(x^2)} = \lim_{x \to 0}\left\{\frac{1 - \frac{x^3}{2} + \frac{x^6}{3} - \frac{x^9}{4} + \cdots}{1 - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \cdots}\right\} = \frac{1-0+0-0+\cdots}{1-0+0-0+\cdots} = \frac{1}{1} = 1$$

---

60) $\sinh^{-1} x = \int_0^x \frac{1}{\sqrt{1+t^2}}\,dt$

use Binomial Series because it is a Taylor Series generated by $f(x) = (1+x)^m$, where m is constant

$\frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-\frac{1}{2}} = 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} (t^2)^k$

$= 1 + \left\{\left(\frac{-1}{2}\right)(t^2)^1 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)}{2!}(t^2)^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)}{3!}(t^2)^3 + \cdots\right\}$

$= 1 + \left\{\frac{-1}{2}t^2 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{(2)}t^4 + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)}{(3)(2)}t^6 + \cdots\right\}$

$= 1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \cdots$

$\sinh^{-1} x \approx \int_0^x \frac{1}{\sqrt{1+t^2}}\,dt = \int_0^x \left(1 - \frac{1}{2}t^2 + \frac{3}{8}t^4 - \frac{5}{16}t^6 + \cdots\right)dt = \left[t - \frac{1}{(3)(2)}t^3 + \frac{3}{(5)(8)}t^5 + \frac{5}{(7)(16)}t^7 + \cdots\right]$

$\approx \left[x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \cdots\right] - [0]$

$\approx x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \cdots$

60) continued

a) 1st four nonzero terms of the Taylor series

$$\sinh^{-1} x \approx x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7$$

b) 1st three terms

$$\sinh^{-1} x \approx x - \frac{1}{6}x^3 + \frac{3}{40}x^5$$

$$\sinh^{-1}(0.25) = \sinh^{-1}\left(\frac{1}{4}\right) \approx \left(\frac{1}{4}\right) - \frac{1}{6}\left(\frac{1}{4}\right)^3 + \frac{3}{40}\left(\frac{1}{4}\right)^5 \approx 0.247469$$

the error is less than the absolute value of the 1st unused term (here it is 4th term), $\left|\frac{-5}{112}x^7\right|$

evaluating at $x = \frac{1}{4} = 0.25$

$$|\text{error}| < \left|\frac{-5}{112}\left(\frac{1}{4}\right)^7\right| \approx 2.725 \times 10^{-6}$$

---

62) $f(x) = \frac{1}{1-x^2} = (1-x^2)^{-1}$ $\quad \frac{df}{dx} = -1(1-x^2)^{-2}(-2x) = \frac{2x}{(1-x^2)^2}$

$$\frac{1}{1-x^2} = 1 + (x^2)^1 + (x^2)^2 + (x^2)^3 + (x^2)^4 + \dots = 1 + x^2 + x^4 + x^6 + x^8 + \dots$$

$$\frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \frac{d}{dx}\left(1 + x^2 + x^4 + x^6 + x^8 + \dots\right)$$

$$= [0] + [2x] + [4x^3] + [6x^5] + [8x^7] + \dots$$

$$= 2x + 4x^3 + 6x^5 + 8x^7 + \dots + (2n)x^{2n-1} + \dots$$

$$= \sum_{n=1}^{\infty} (2n)x^{2n-1}$$