Theorem 23 - Taylor's Theorem

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(a)}{(n+1)!}(b-a)^{n+1}.$$

Taylor's Formula

Let f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \tag{1}$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } a \text{ and } x. \tag{2}$$

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k .$$

Theorem 24 - The Remainder Estimation Theorem

If there is a positive constant M such that $\left|f^{(n+1)}(t)\right| \le M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$\left|R_n(x)\right| \le M \frac{\left|x-a\right|^{n+1}}{(n+1)!}.$$

If this inequality hold for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

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2) $e^{-\frac{x}{2}}$ $\quad e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-\frac{x}{2}} = 1 + \left(\frac{-x}{2}\right) + \frac{\left(\frac{-x}{2}\right)^2}{2!} + \frac{\left(\frac{-x}{2}\right)^3}{3!} + \cdots = 1 + \frac{(-1)^1(x)^1}{(2^1)1!} + \frac{(-1)^2(x)^2}{(2^2)2!} + \frac{(-1)^3(x)^3}{(2^3)3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!}$$

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4) $\sin\left(\frac{\pi x}{2}\right)$ $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1) x^{2n+1}}{(2n+1)!}$

$$\sin\left(\frac{\pi x}{2}\right) = \left(\frac{\pi x}{2}\right) - \frac{\left(\frac{\pi x}{2}\right)^3}{3!} + \frac{\left(\frac{\pi x}{2}\right)^5}{5!} + \cdots$$

$$= \frac{(-1)^0(\pi^1)x^1}{(2^1)\,1!} + \frac{(-1)^1(\pi^3)x^3}{(2^3)\,3!} + \frac{(-1)^2(\pi^5)x^5}{(2^5)\,5!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(\pi^{2n+1})x^{2n+1}}{(2^{2n+1})\,(2n+1)!}$$

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6) $\cos\left(\frac{x^{2/3}}{\sqrt{2}}\right)$ $\quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\cos\left(\frac{x^{2/3}}{\sqrt{2}}\right) = 1 - \frac{\left(\frac{x^{2/3}}{\sqrt{2}}\right)^2}{2!} + \frac{\left(\frac{x^{2/3}}{\sqrt{2}}\right)^4}{4!} + \cdots$$

$$= 1 - \frac{(-1)^1(x^{2/3})^2}{(\sqrt{2})^2\, 2!} + \frac{(-1)^2(x^{2/3})^4}{(\sqrt{2})^4\, 4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^{2/3})^{2n}}{(\sqrt{2})^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{4}{3}n}}{(2^n)(2n)!}$$

8) $\tan^{-1}(3x^4)$ $\quad \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$$\tan^{-1}(3x^4) = (3x^4) - \frac{(3x^4)^3}{3} + \frac{(3x^4)^5}{5} - \cdots$$

$$= (-1)^0(3^1)(x^4)^1 + \frac{(-1)^1(3^3)(x^4)^3}{3} + \frac{(-1)^2(3^5)(x^4)^5}{5} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (3^{2n+1})(x^4)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n (3^{2n+1}) x^{8n+4}}{2n+1}$$

---

10) $\frac{1}{2-x}$ $\quad \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$

$$\frac{1}{2-x} = \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right) = \frac{1}{2}\left\{1 + \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 + \cdots\right\}$$

$$= \frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \cdots = \frac{x^0}{2^1} + \frac{x^1}{2^2} + \frac{x^2}{2^3} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

---

12) $e^{-x^2+\ln 5}$ $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-x^2+\ln 5} = e^{(\ln 5 - x^2)} = (e^{\ln 5})(e^{-x^2}) = 5e^{-x^2} = 5\left\{1 + (-x^2)^1 + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots\right\}$$

$$= 5\left\{(-1)^0(x^2)^0 + (-1)^1(x^2)^1 + \frac{(-1)^2(x^2)^2}{2!} + \frac{(-1)^3(x^2)^3}{3!} + \cdots\right\}$$

$$= 5\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^n}{n!} = 5\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

14) $x^2 \sin x$ $\qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$x^2 \sin x = x^2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) = x^2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!} = x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} + \cdots$$

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16) $\sin x - x + \frac{x^3}{3!}$ $\qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin x - x + \frac{x^3}{3!} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right) - x + \frac{x^3}{3!} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \right) - x + \frac{x^3}{3!}$$

$$= \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=2}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

---

18) $x^2 \cos(x^2)$ $\qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$x^2 \cos(x^2) = x^2 \left(\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} \right) = x^2 \left(1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \cdots \right)$$

$$= x^2 - \frac{x^6}{2!} + \frac{x^{10}}{4!} - \cdots = (-1)^0 x^{0+2} + \frac{(-1)^1 x^{4+2}}{2!} + \frac{(-1)^2 x^{8+2}}{4!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n} (x^2)}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n)!}$$

---

20) $\sin^2 x$ $\qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos(2x)\right) = \frac{1}{2}\left(1 - \left(\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} \right)\right)$$

$$= \frac{1}{2}\left(1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots \right)\right) = \frac{1}{2}\left(\frac{(2^2)x^2}{2!} - \frac{(2^4)x^4}{4!} + \cdots \right)$$

$$= \frac{(-1)^0 (2^2) x^2}{(2)\,2!} + \frac{(-1)^1 (2^4) x^4}{(2)\,4!} + \cdots = \frac{(-1)^0 (2^1) x^2}{2!} + \frac{(-1)^1 (2^3) x^4}{4!} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2^{2n-1}) x^{2n}}{(2n)!} \quad \text{or} \quad = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2^{2n-1}) x^{2n}}{(2n)!}$$

22) $x \ln(1+2x)$ $\qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

$$x \ln(1+2x) = x\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2x)^n}{n}\right) = x\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2^n)x^n}{n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2^n)(x^n)(x^1)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2^n)x^{n+1}}{n}$$

---

24) $\frac{2}{(1-x)^3}$ $\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

let $f(x) = \frac{1}{1-x} = (1-x)^{-1}$ $\quad \frac{df}{dx} = [-1(1-x)^{-2}(-1)] = (1-x)^{-2} = \frac{1}{(1-x)^2}$

$$\frac{d^2f}{dx^2} = [-2(1-x)^{-3}(-2)] = \frac{2}{(1-x)^3}$$

$$\frac{2}{(1-x)^3} = \frac{d^2}{dx}\left(\frac{1}{1-x}\right) = \frac{d^2}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} (n)x^{n-1}\right)$$

$$= \sum_{n=2}^{\infty} (n)(n-1)x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)x^n$$

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26) $(\sin x)(\cos x)$ $\qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin x \cos x = \frac{1}{2}\sin(2x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(-1)^n (2^{2n+1}) x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n (2^{2n})(2^1) x^{2n+1}}{2(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2^{2n}) x^{2n+1}}{(2n+1)!}$$

28) $\cos x - \sin x$ $\qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\cos x - \sin x = \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\right) - \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n x^{2n}}{(2n)!} - \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right)$$

---

30) $\ln(1+x) - \ln(1-x)$ $\qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

$$\ln(1+x) - \ln(1-x) = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}\right) - \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} (-x)^n}{n}\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \left((-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \cdots\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots = \frac{2x^1}{1} + \frac{2x^{2+1}}{2+1} + \frac{2x^{4+1}}{4+1} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$$

---

32) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$\frac{\ln(1+x)}{1-x} = (\ln(1-x))\left(\frac{1}{1-x}\right)$$

32) continued

$$\frac{\ln(1+x)}{1-x} = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{12}x^4 + \cdots$$

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34) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos^2 x \sin x = \cos x (\sin x \cos x) = \cos x \left(\tfrac{1}{2}\sin(2x)\right)$$

$$= \tfrac{1}{2}\cos x \sin(2x) = \tfrac{1}{2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)\left((2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots\right)$$

$$= x - \frac{7}{6}x^3 + \frac{61}{120}x^5 - \frac{1247}{5040}x^7 + \cdots$$

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36) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

$$\sin(\tan^{-1} x) = \sin\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)$$

$$= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) - \frac{1}{3!}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)^3$$

$$+ \frac{1}{5!}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) - \frac{1}{7!}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) + \cdots$$

$$\vdots$$

$$= x - \frac{1}{2}x^3 + \frac{3}{8}x^5 - \frac{5}{16}x^7 + \cdots$$

38) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\cos\sqrt{x} + \ln(\cos x) = \left(1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \cdots\right) + \ln\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$= \left(1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{6!} + \cdots\right) + \left\{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) - \frac{1}{2}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^2\right.$$

$$\left. + \frac{1}{3}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^3 - \frac{1}{4}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)^4 + \cdots\right\}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} - \frac{1}{2}x - \frac{11}{24}x^2 - \frac{1}{720}x^3 - \cdots = 1 - \frac{1}{2}x - \frac{11}{24}x^2 - \frac{1}{720}x^3 - \cdots$$

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40) $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

since $n=4$, $f^{(5)}(x) = e^x$ and $|f^{(5)}(x)| \le M$ on $[0, \frac{1}{2}]$

$\Downarrow$

$|e^x| \le e^{\frac{1}{2}} = \sqrt{e}$ on $[0, \frac{1}{2}]$

$\Downarrow$

let $M = 2.7$

5th term

$\downarrow$

then $|R_4(0.5)| \le M \frac{|(0.5)-(0)|^5}{5!} = (2.7)\frac{(0.5)^5}{5!} \approx 7.03 \times 10^{-4}$

error $\le 7.03 \times 10^{-4}$