Definition

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by** f **at** $x=a$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots.$$

The **Maclaurin series of** f is the Taylor series generated by f at $x=0$, or

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots.$$

Definition

Let f be a function with derivatives of order k for $k=1,2,\dots,N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order** n generated by f at $x=a$ is the polynomial

$$P_n(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Table 10.1 Frequently Used Taylor Series (from section 10.10)

$$\frac{1}{1-x}=1+x+x^2+\cdots+x^n+\cdots=\sum_{n=0}^{\infty}x^n \qquad |x|<1$$

$$\frac{1}{1+x}=1-x+x^2-\cdots+(-x)^n+\cdots=\sum_{n=0}^{\infty}(-1)^n x^n \qquad |x|<1$$

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad |x|<\infty$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad |x|<\infty$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \qquad |x|<\infty$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \qquad -1<x\le 1$$

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+(-1)^n\frac{x^{2n+1}}{2n+1}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \qquad |x|\le 1$$

2) $f(x) = \sin x \quad , a=0 \qquad f(0) = \sin(0) = 0$

$f'(x) = \frac{df}{dx} = \cos x\,(1) = \cos x \qquad f'(0) = \cos(0) = 1$

$f''(x) = \frac{d^2f}{dx^2} = -\sin x\,(1) = -\sin x \qquad f''(0) = -\sin(0) = -(0) = 0$

$f'''(x) = \frac{d^3f}{dx^3} = -\cos(x)(1) = -\cos x \qquad f'''(0) = -\cos(0) = -1$

$P_0(x) = f(0) = 0 \qquad P_1(x) = f(0) + f'(0)(x-(0))^1 = 0 + 1(x)^1 = x$

$P_2(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 = 0 + 1(x)^1 + \frac{0}{2}(x)^2 = x$

$P_3(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 + \frac{f'''(0)}{3!}(x-(0))^3$

$= 0 + 1(x)^1 + \frac{1}{2}(x)^2 + -\frac{1}{6}(x)^3 = x - \frac{1}{6}x^3$

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4) $f(x) = \ln(1+x) \quad , a=0 \qquad f(0) = \ln(1+(0)) = \ln 1 = 0$

$f'(x) = \frac{df}{dx} = \frac{1}{1+x}(1) = \frac{1}{1+x} = (1+x)^{-1} \qquad f'(0) = \frac{1}{1+(0)} = \frac{1}{1} = 1$

$f''(x) = \frac{d^2f}{dx^2} = -1(1+x)^{-2}(1) = \frac{-1}{(1+x)^2} \qquad f''(0) = \frac{-1}{(1+(0))^2} = \frac{-1}{1^2} = -1$

$f'''(x) = \frac{d^3f}{dx^3} = -1\left[-2(1+x)^{-3}(1)\right] = \frac{2}{(1+x)^3} \qquad f'''(0) = \frac{2}{(1+(0))^3} = \frac{2}{1^3} = 2$

$P_0(x) = f(0) = 0 \qquad P_1(x) = f(0) + f'(0)(x-(0))^1 = 0 + 1(x)^1 = x$

$P_2(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 = 0 + 1(x)^1 + \frac{(-1)}{2}(x)^2 = x - \frac{1}{2}x^2$

$P_3(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 + \frac{f'''(0)}{3!}(x-(0))^3$

$= 0 + 1(x)^1 + \frac{(-1)}{2}(x)^2 + \frac{(2)}{(2)(3)}(x)^3 = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

6) $f(x) = \frac{1}{x+2} = (x+2)^{-1}$, $a = 0$ $\quad f(0) = \frac{1}{(0)+2} = \frac{1}{2}$

$f'(x) = -1(x+2)^{-2}(1) = -1(x+2)^{-2} = \frac{-1}{(x+2)^2}$ $\quad f'(0) = \frac{-1}{((0)+2)^2} = \frac{-1}{4}$

$f''(x) = -1\left[-2(x+2)^{-3}(1)\right] = 2(x+2)^{-3} = \frac{2}{(x+2)^3}$ $\quad f''(0) = \frac{2}{((0)+2)^3} = \frac{2}{(2)^3} = \frac{1}{4}$

$f'''(x) = 2\left[-3(x+2)^{-4}(1)\right] = \frac{-6}{(x+2)^4}$ $\quad f'''(0) = \frac{-6}{((0)+2)^4} = \frac{-6}{(2)^4} = \frac{-3}{(2)^3} = \frac{-3}{8}$

$P_0(x) = f(0) = \frac{1}{2}$ $\quad P_1(x) = f(0) + f'(0)(x-(0))^1 = \frac{1}{2} + \left(\frac{-1}{4}\right)(x)^1 = \frac{1}{2} - \frac{1}{4}x$

$P_2(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 = \frac{1}{2} + \left(\frac{-1}{4}\right)(x)^1 + \frac{\left(\frac{1}{4}\right)}{2}(x)^2 = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2$

$P_3(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 + \frac{f'''(0)}{3!}(x-(0))^3$

$= \frac{1}{2} + \left(\frac{-1}{4}\right)(x)^1 + \frac{\left(\frac{1}{4}\right)}{2}(x)^2 + \frac{\left(\frac{-3}{8}\right)}{(2)(3)}(x)^3 = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3$

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8) $f(x) = \tan x$, $a = \frac{\pi}{4}$ $\quad f\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \frac{df}{dx} = \sec^2 x(1) = \sec^2 x$ $\quad f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = (\sqrt{2})^2 = 2$

$f''(x) = \frac{d^2f}{dx^2} = 2\sec x(\sec x \tan x) = 2\sec^2 x \tan x$ $\quad f''\left(\frac{\pi}{4}\right) = 2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) = 2(\sqrt{2})^2(1) = 4$

$f'''(x) = \frac{d^3f}{dx^3} = (2\sec^2 x)\left[\sec^2 x(1)\right] + (\tan x)\left[4\sec x(\sec x \tan x(1))\right] = 2\sec^4 x + 4\sec^2 x \tan^2 x$

$f'''\left(\frac{\pi}{4}\right) = 2(\sqrt{2})^4 + 4(\sqrt{2})^2(1)^2 = 2(2)^2 + 4(2) = 8 + 8 = 16$

$P_0(x) = f\left(\frac{\pi}{4}\right) = 1$ $\quad P_1(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \left(\frac{\pi}{4}\right)\right)^1 = 1 + 2\left(x - \frac{\pi}{4}\right)^1 = 1 + 2\left(x - \frac{\pi}{4}\right)$

$P_2(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \left(\frac{\pi}{4}\right)\right)^1 + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \left(\frac{\pi}{4}\right)\right)^2 = 1 + 2\left(x - \frac{\pi}{4}\right)^1 + \frac{(4)}{2}\left(x - \frac{\pi}{4}\right)^2$

$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2$

$P_3(x) = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \left(\frac{\pi}{4}\right)\right)^1 + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x - \left(\frac{\pi}{4}\right)\right)^2 + \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x - \left(\frac{\pi}{4}\right)\right)^3$

$= 1 + 2\left(x - \frac{\pi}{4}\right)^1 + \frac{(4)}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{(16)}{(2)(3)}\left(x - \frac{\pi}{4}\right)^3$

$= 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$

10) $f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$, $a = 0$ $\qquad f(0) = \sqrt{1-(0)} = \sqrt{1} = 1$

$$f'(x) = \frac{df}{dx} = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) = \frac{-1}{2}(1-x)^{\frac{-1}{2}} = \frac{-1}{2\sqrt{1-x}} \qquad f'(0) = \frac{-1}{2\sqrt{1-(0)}} = \frac{-1}{2}$$

$$f''(x) = \frac{d^2f}{dx^2} = \frac{-1}{2}\left[\frac{-1}{2}(1-x)^{\frac{-3}{2}}(-1)\right] = \frac{-1}{4}(1-x)^{\frac{-3}{2}} = \frac{-1}{4(\sqrt{1-x})^3} \qquad f''(0) = \frac{-1}{4(\sqrt{1-(0)})^3} = \frac{-1}{4}$$

$$f'''(x) = \frac{d^3f}{dx^3} = \frac{-1}{4}\left[\frac{-3}{2}(1-x)^{\frac{-5}{2}}(-1)\right] = \frac{-3}{8}(1-x)^{\frac{-5}{2}} = \frac{-3}{8(\sqrt{1-x})^5} \qquad f'''(0) = \frac{-3}{8(\sqrt{1-(0)})^5} = \frac{-3}{8}$$

$$P_0(x) = f(0) = 1 \qquad P_1(x) = f(0) + f'(0)(x-(0))^1 = 1 + \left(\frac{-1}{2}\right)(x)^1 = 1 - \frac{1}{2}x$$

$$P_2(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 = 1 + \left(\frac{-1}{2}\right)(x)^1 + \frac{\left(\frac{-1}{4}\right)}{2}(x)^2 = 1 - \frac{1}{2}x - \frac{1}{8}x^2$$

$$P_3(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 + \frac{f'''(0)}{3!}(x-(0))^3$$

$$= 1 + \left(\frac{-1}{2}\right)(x)^1 + \frac{\left(\frac{-1}{4}\right)}{2}(x)^2 + \frac{\left(\frac{-3}{8}\right)}{(2)(3)}(x)^3 = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$$

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12) $f(x) = xe^x \qquad f(0) = (0)e^{(0)} = (0)(1) = 0$

$$f'(x) = \frac{df}{dx} = (x)[e^x(1)] + (e^x)[1] = xe^x + e^x \qquad f'(0) = (0)e^{(0)} + e^{(0)} = 0 + 1 = 1$$

$$f''(x) = \frac{d^2f}{dx^2} = \{(x)[e^x(1)] + (e^x)[1]\} + [e^x(1)] = xe^x + 2e^x$$

$$f''(0) = (0)e^{(0)} + 2e^{(0)} = 0 + 2(1) = 2$$

$$f'''(x) = \frac{d^3f}{dx^3} = \{(x)[e^x(1)] + (e^x)[1]\} + 2[e^x(1)] = xe^x + 3e^x$$

$$f'''(0) = (0)e^{(0)} + 3e^{(0)} = 0 + 3(1) = 3$$

$\vdots$

$$f^{(k)}(x) = \frac{d^kf}{dx^k} = \{(x)[e^x(1)] + (e^x)[1]\} + (k-1)[e^x(1)] = xe^x + ke^x$$

$$f^{(k)}(0) = (0)e^{(0)} + ke^{(0)} = 0 + k(1) = k$$

$$xe^x = f(0) + f'(0)(x)^1 + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \dots + \frac{f^{(k)}(0)}{k!}(x)^k + \dots$$

$$= (0) + (1)(x)^1 + \frac{(2)}{(1)(2)}(x)^2 + \frac{(3)}{(1)(2)(3)}(x)^3 + \dots + \frac{(k)}{(1)(2)\cdots(k-1)(k)}(x)^k + \dots$$

$$= (0) + x + x^2 + \frac{1}{2!}x^3 + \dots + \frac{1}{(k-1)!}x^k + \dots = \sum_{k=0}^{\infty} \frac{1}{(k-1)!}x^k$$

14) $f(x) = \dfrac{2+x}{1-x}$ $\qquad f(0) = \dfrac{2+(0)}{1-(0)} = \dfrac{2}{1} = 2$

$$f'(x) = \frac{df}{dx} = \frac{(1-x)[1] - (2+x)[-1]}{(1-x)^2} = \frac{1-x+2+x}{(1-x)^2} = \frac{3}{(1-x)^2} = 3(1-x)^{-2}$$

$$f'(0) = \frac{3}{(1-(0))^2} = \frac{3}{(1)^2} = \frac{3}{1} = 3 = 3(1)$$

$$f''(x) = \frac{d^2f}{dx^2} = 3\left[-2(1-x)^{-3}(-1)\right] = 6(1-x)^{-3} = \frac{6}{(1-x)^3} \qquad f''(0) = \frac{6}{(1-(0))^3} = \frac{6}{1} = 6 = 3(2)$$

$$f'''(x) = \frac{d^3f}{dx^2} = 6\left[-3(1-x)^{-4}(-1)\right] = 18(1-x)^{-4} = \frac{18}{(1-x)^4} \qquad f'''(0) = \frac{18}{(1-(0))^4} = \frac{18}{1} = 18 = 3(6)$$

$$\vdots$$

$$f^{(k)}(x) = \frac{d^kf}{dx^k} = \frac{3(k!)}{(1-x)^{(k+1)}} \qquad f^{(k)}(0) = \frac{3(k!)}{(1-(0))^{(k+1)}} = \frac{3(k!)}{1} = 3(k!)$$

$$\frac{2+x}{1-x} = f(0) + f'(0)(x)^1 + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \cdots + \frac{f^{(k)}(0)}{k!}(x)^k + \cdots$$

$$= 2 + (3)(x)^1 + \frac{(6)}{(2)}(x)^2 + \frac{(18)}{(2)(3)}(x)^3 + \cdots + \frac{3(k!)}{k!}(x)^k + \cdots$$

$$= 2 + 3x^1 + 3x^2 + 3x^3 + \cdots + 3x^k + \cdots$$

$$= 2 + \sum_{k=1}^{\infty} 3x^k$$

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16) $f(x) = \sin\left(\frac{x}{2}\right)$ $\qquad f(0) = \sin\left(\frac{0}{2}\right) = 0$

$$f'(x) = \frac{df}{dx} = \left[\cos\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)\right] = \frac{1}{2}\cos\left(\frac{x}{2}\right) \qquad f'(0) = \frac{1}{2}\cos\left(\frac{0}{2}\right) = \frac{1}{2}(1) = \frac{1}{2}$$

$$f''(x) = \frac{d^2f}{dx^2} = \frac{1}{2}\left[-\sin\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)\right] = \frac{-1}{4}\sin\left(\frac{x}{2}\right) \qquad f''(0) = \frac{-1}{4}\sin\left(\frac{0}{2}\right) = \frac{-1}{4}(0) = 0$$

$$f'''(x) = \frac{d^3f}{dx^3} = \frac{-1}{4}\left[\cos\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)\right] = \frac{-1}{8}\cos\left(\frac{x}{2}\right) \qquad f'''(0) = \frac{-1}{8}\cos\left(\frac{0}{2}\right) = \frac{-1}{8}(1) = \frac{-1}{8}$$

$$f^{(4)}(x) = \frac{d^4f}{dx^4} = \frac{-1}{8}\left[-\sin\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)\right] = \frac{1}{16}\sin\left(\frac{x}{2}\right) \qquad f^{(4)}(0) = \frac{1}{16}\sin\left(\frac{0}{2}\right) = \frac{1}{16}(0) = 0$$

$$f^{(5)}(x) = \frac{d^5f}{dx^5} = \frac{1}{16}\left[\cos\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)\right] = \frac{1}{32}\cos\left(\frac{x}{2}\right) \qquad f^{(5)}(0) = \frac{1}{32}\cos\left(\frac{0}{2}\right) = \frac{1}{32}(1) = \frac{1}{32}$$

$$\vdots$$

16) continued

$$\sin\left(\frac{x}{2}\right) = f(0) + f'(0)(x)^1 + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \frac{f^{(4)}(0)}{4!}(x)^4 + \frac{f^{(5)}(0)}{5!}(x)^5 + \ldots$$

$$= (0) + \left(\frac{1}{2}\right)(x)^1 + \frac{(0)}{2!}(x)^2 + \frac{\left(\frac{-1}{8}\right)}{3!}(x)^3 + \frac{(0)}{4!}(x)^4 + \frac{\left(\frac{1}{32}\right)}{5!}(x)^5 + \ldots$$

$$= \frac{1}{2}x - \frac{1}{8}\left(\frac{1}{3!}\right)x^3 + \frac{1}{32}\left(\frac{1}{5!}\right)x^5 + \ldots$$

$$= (-1)^0\left(\frac{1}{2}\right)x^1 + (-1)^1\left(\frac{1}{2^3}\right)\left(\frac{1}{3!}\right)x^3 + (-1)^2\left(\frac{1}{2^5}\right)\left(\frac{1}{5!}\right)x^5 + \ldots + (-1)^k\left(\frac{1}{2^{(2k+1)}}\right)\left(\frac{1}{(2k+1)!}\right)x^{(2k+1)} + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)}}{2^{(2k+1)}(2k+1)!}$$

"better method"

from Table 10.1 in section 10.10, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$, $|x| < \infty$

$$\sin\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{(2n+1)}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{x^{(2n+1)}}{2^{(2n+1)}}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{2^{(2n+1)}(2n+1)!}$$

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18) $f(x) = 5\cos(\pi x)$ $\quad$ $f(0) = 5\cos(\pi(0)) = 5(1) = 5$

$$f'(x) = \frac{df}{dx} = 5[-\sin(\pi x)(\pi)] = -5\pi\sin(\pi x)$$

$$f'(0) = -5\pi\sin(\pi(0)) = -5\pi(0) = 0$$

$$f''(x) = \frac{d^2f}{dx^2} = -5\pi[\cos(\pi x)(\pi)] = -5\pi^2\cos(\pi x)$$

$$f''(0) = -5\pi^2\cos(\pi(0)) = -5\pi^2(1) = -5\pi^2$$

$$f'''(x) = \frac{d^3f}{dx^2} = -5\pi^2[-\sin(\pi x)(\pi)] = 5\pi^3\sin(\pi x)$$

$$f'''(0) = 5\pi^3\sin(\pi(0)) = 5\pi^3(0) = 0$$

$$f^{(4)}(x) = \frac{d^4f}{dx^4} = 5\pi^3[\cos(\pi x)(\pi)] = 5\pi^4\cos(\pi x)$$

$$f^{(4)}(0) = 5\pi^4\cos(\pi(0)) = 5\pi^4(1) = 5\pi^4$$

18) continued

$$5\cos(\pi x) = f(0) + f'(0)(x)^1 + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \frac{f^{(4)}(0)}{4!}(x)^4 + \dots$$

$$= (5) + (0)(x)^1 + \frac{(-5\pi^2)}{2!}(x)^2 + \frac{(0)}{3!}(x)^3 + \frac{(5\pi^4)}{4!}(x)^4 + \dots$$

$$= 5 - 5\frac{\pi^2}{2!}x^2 + 5\frac{\pi^4}{4!}x^4 + \dots = 5\left\{1 - \frac{\pi^2}{2!}x^2 + \frac{\pi^4}{4!}x^4 + \dots\right\}$$

$$= 5\left\{(-1)^0 + (-1)^1\frac{\pi^2}{2!}x^2 + (-1)^2\frac{\pi^4}{4!}x^4 + \dots\right\} = 5\sum_{k=0}^{\infty}(-1)^k\frac{\pi^{2k}}{(2k)!}x^{2k}$$

"better method"

from Table 10.1 in section 10.10, $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

$$5\cos(\pi x) = 5\sum_{n=0}^{\infty}\frac{(-1)^n(\pi x)^{2n}}{(2n)!} = 5\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}x^{2n}}{(2n)!} = 5\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{(2n)!}x^{2n}$$

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20) $\sinh x = \frac{e^x - e^{-x}}{2}$

From Table 10.1 in section 10.10, $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad |x| < \infty$

$$e^{-x} = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left\{e^x - e^{-x}\right\} = \frac{1}{2}\left\{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)\right\}$$

$$= \frac{1}{2}\left\{2(x) + 2\left(\frac{x^3}{3!}\right) + \dots\right\} = x + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty}\frac{x^{(2n+1)}}{(2n+1)!}$$

22) $f(x) = \frac{x^2}{x+1}$ $\qquad f(0) = \frac{(0)^2}{(0)+1} = 0$

$$f'(x) = \frac{df}{dx} = \frac{(x+1)[2x] - (x^2)[1]}{(x+1)^2} = \frac{2x^2+2x-x^2}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2}$$

$$f'(0) = \frac{(0)^2+2(0)}{((0)+1)^2} = 0$$

$$f''(x) = \frac{d^2f}{dx^2} = \frac{((x+1)^2)[2x+2] - (x^2+2x)[2(x+1)'(1)]}{((x+1)^2)^2} = \frac{(x+1)\{(x+1)[2x+2] - (x^2+2x)[2]}{(x+1)^4}$$

$$= \frac{(2x^2+4x+2) - (2x^2+4x)}{(x+1)^3} = \frac{2}{(x+1)^3} = 2(x+1)^{-3}$$

$$f''(0) = \frac{2}{((0)+1)^3} = 2$$

$$f'''(x) = \frac{d^3f}{dx^3} = 2[-3(x+1)^{-4}(1)] = -6(x+1)^{-4} = \frac{-6}{(x+1)^4} \qquad f'''(0) = \frac{-6}{((0)+1)^4} = -6$$

$$f^{(4)}(x) = \frac{d^4f}{dx^4} = -6[-4(x+1)^{-5}(1)] = \frac{24}{(x+1)^5} \qquad f^{(4)}(0) = \frac{24}{((0)+1)^5} = 24$$

$$\vdots$$

$$f^{(k)}(x) = \frac{d^kf}{dx^k} = \frac{(-1)^k k!}{(x+1)^{k+1}} \text{ for } k \geq 2 \qquad f^{(k)}(0) = \frac{(-1)^k k!}{((0)+1)^{k+1}} = (-1)k!$$

$$\frac{x^2}{x+1} = f(0) + f'(0)(x)^1 + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \frac{f^{(4)}(0)}{4!}(x)^4 + \dots + \frac{f^{(k)}(0)}{k!}(x)^k + \dots$$

$$= (0) + (0)(x)^1 + \frac{(2)}{2!}(x)^2 + \frac{(-6)}{3!}(x)^3 + \frac{(24)}{4!}(x)^4 + \dots + \frac{(-1)^k k!}{k!}(x)^k + \dots$$

$$= x^2 - x^3 + x^4 + \dots + (-1)^k x^k + \dots$$

$$= (-1)^2 x^2 + (-1)^3 x^3 + (-1)^4 x^4 + \dots + (-1)^k x^k + \dots$$

$$= \sum_{k=2}^{\infty} (-1)^k x^k$$

24) $f(x)=(x+1)\ln(x+1)$ $\qquad f(0)=((0)+1)\ln((0)+1)=(1)\ln 1=(1)(0)=0$

$$f'(x)=\frac{df}{dx}=(x+1)\left[\frac{1}{x+1}(1)\right]+(\ln(x+1))[1]=1+\ln(x+1)$$

$$f'(0)=1+\ln((0)+1)=1+(0)=1$$

$$f''(x)=\frac{d^2f}{dx^2}=\left[\frac{1}{x+1}(1)\right]=\frac{1}{x+1}=(x+1)^{-1} \qquad f''(0)=\frac{1}{(0)+1}=\frac{1}{1}=1$$

$$f'''(x)=\frac{d^3f}{dx^3}=-1(x+1)^{-2}(1)=-1(x+1)^{-2}=\frac{-1}{(x+1)^2} \qquad f'''(0)=\frac{-1}{((0)+1)^2}=\frac{-1}{1^2}=-1$$

$$f^{(4)}(x)=\frac{d^4f}{dx^4}=-1\left[-2(x+1)^{-3}(1)\right]=2(x+1)^{-3}=\frac{2}{(x+1)^3} \qquad f^{(4)}(0)=\frac{2}{((0)+1)^3}=\frac{2}{1^3}=2$$

$$f^{(5)}(x)=\frac{d^5f}{dx^5}=2\left[-3(x+1)^{-4}(1)\right]=-6(x+1)^{-4}=\frac{-6}{(x+1)^4} \qquad f^{(5)}(0)=\frac{-6}{((0)+1)^4}=\frac{-6}{1^4}=-6$$

$$\vdots$$

$$f^{(k)}(x)=\frac{d^kf}{dx^k}=\frac{(-1)^k(k-2)!}{(x+1)^{(k-1)}} \text{ for } k\geq 2 \qquad f^{(k)}(0)=\frac{(-1)^k(k-2)!}{((0)+1)^{(k-1)}}=(-1)^k(k-2)!$$

$$(x+1)\ln(x+1)=f(0)+f'(0)(x)^1+\frac{f''(0)}{2!}(x)^2+\frac{f'''(0)}{3!}(x)^3+\frac{f^{(4)}(0)}{4!}(x)^4+\frac{f^{(5)}(0)}{5!}(x)^5+\dots+\frac{f^{(k)}(0)}{k!}(x)^k+\dots$$

$$=(0)+(1)(x)^1+\frac{(1)}{2!}(x)^2+\frac{(-1)}{3!}(x)^3+\frac{(2)}{4!}(x)^4+\frac{(-6)}{5!}(x)^5+\dots+\frac{(-1)^k(k-2)!}{k!}(x)^k+\dots$$

$$=x+\frac{1}{2}x^2-\frac{1}{(2)(3)}x^3+\frac{1}{(3)(4)}x^4-\frac{1}{(4)(5)}x^5+\dots+\frac{(-1)^k}{(k-1)(k)}x^k+\dots$$

$$=x+\left\{\frac{(-1)^2}{(1)(2)}x^2+\frac{(-1)^3}{(2)(3)}x^3+\frac{(-1)^4}{(3)(4)}x^4+\frac{(-1)^5}{(4)(5)}x^5+\dots+\frac{(-1)^k}{(k-1)(k)}x^k+\dots\right\}$$

$$=x+\sum_{k=2}^{\infty}\frac{(-1)^k x^k}{(k-1)k}$$

26) $f(x) = 2x^3 + x^2 + 3x - 8$, $a = 1$

$f(1) = 2(1)^3 + (1)^2 + 3(1) - 8 = 6 - 8 = -2$

$f'(x) = \frac{df}{dx} = 6x^2 + 2x + 3$ $\quad f'(1) = 6(1)^2 + 2(1) + 3 = 11$

$f''(x) = \frac{d^2f}{dx^2} = 12x + 2$ $\quad f''(1) = 12(1) + 2 = 14$

$f'''(x) = \frac{d^3f}{dx^3} = 12$ $\quad f'''(1) = 12$

$f^{(4)}(x) = \frac{d^4f}{dx^4} = 0$ $\quad f^{(4)}(1) = 0$

$\vdots$

$f^{(k)}(x) = \frac{d^kf}{dx^k} = 0$ for $k \geq 4$ $\quad f^{(k)}(1) = 0$

$$T(x) = f(1) + f'(1)(x-(1))^1 + \frac{f''(1)}{2!}(x-(1))^2 + \frac{f'''(1)}{3!}(x-(1))^3 + \frac{f^{(4)}(1)}{4!}(x-(1))^4 + \cdots$$

$$= (-2) + (11)(x-1)^1 + \frac{(14)}{2}(x-1)^2 + \frac{(12)}{(2)(3)}(x-1)^3 + \frac{(0)}{4!}(x-1)^4 + \cdots$$

$$= -2 + 11(x-1) + 7(x-1)^2 + 2(x-1)^3$$

---

28) $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2$, $a = -1$

$f(-1) = 3(-1)^5 - (-1)^4 + 2(-1)^3 + (-1)^2 - 2 = -3 - 1 - 2 + 1 - 2 = -7$

$f'(x) = \frac{df}{dx} = 15x^4 - 4x^3 + 6x^2 + 2x$

$f'(-1) = 15(-1)^4 - 4(-1)^3 + 6(-1)^2 + 2(-1) = 15 + 4 + 6 - 2 = 23$

$f''(x) = \frac{d^2f}{dx^2} = 60x^3 - 12x^2 + 12x + 2$

$f''(-1) = 60(-1)^3 - 12(-1)^2 + 12(-1) + 2 = -60 - 12 - 12 + 2 = -82$

28) continued

$f'''(x) = \frac{d^3f}{dx^3} = 180x^2 - 24x + 12 \qquad f'''(-1) = 180(-1)^2 - 24(-1) + 12 = 180 + 24 + 12 = 216$

$f^{(4)}(x) = \frac{d^4f}{dx^4} = 360x - 24 \qquad f^{(4)}(-1) = 360(-1) - 24 = -360 - 24 = -384$

$f^{(5)}(x) = \frac{d^5f}{dx^5} = 360 \qquad f^{(5)}(-1) = 360$

$\vdots$

$f^{(k)}(x) = \frac{d^kf}{dx^k} = 0$ for $k \geq 6$ $\quad f^{(k)}(-1) = 0$

$$T(x) = f(-1) + f'(-1)(x-(-1))^1 + \frac{f''(-1)}{2!}(x-(-1))^2 + \frac{f'''(-1)}{3!}(x-(-1))^3$$

$$+ \frac{f^{(4)}(-1)}{4!}(x-(-1))^4 + \frac{f^{(5)}(-1)}{5!}(x-(-1))^5 + \frac{f^{(k)}(-1)}{k!}(x-(-1))^k \text{ for } k \geq 6$$

$$= (-7) + (23)(x+1)^1 + \frac{(-82)}{2}(x+1)^2 + \frac{(216)}{(2)(3)}(x+1)^3 + \frac{(-384)}{(2)(3)(4)}(x+1)^4$$

$$+ \frac{(360)}{(2)(3)(4)(5)}(x+1)^5 + \frac{(0)}{k!}(x+1)^k \text{ for } k \geq 6$$

$$= -7 + 23(x+1) - 41(x+1)^2 + 36(x+1)^3 - 16(x+1)^4 + 3(x+1)^5$$

---

30) $f(x) = \frac{1}{(1-x)^3} = (1-x)^{-3} \quad a = 0 \quad f(0) = \frac{1}{(1-(0))^3} = \frac{1}{1^3} = 1$

$f'(x) = \frac{df}{dx} = -3(1-x)^{-4}(-1) = 3(1-x)^{-4} = \frac{3}{(1-x)^4} \quad f'(0) = \frac{3}{(1-(0))^4} = \frac{3}{1^4} = 3$

$f''(x) = \frac{d^2f}{dx^2} = 3[-4(1-x)^{-5}(-1)] = 12(1-x)^{-5} = \frac{12}{(1-x)^5} \quad f''(0) = \frac{12}{(1-(0))^5} = \frac{12}{1^5} = 12$

$f'''(x) = \frac{d^3f}{dx^3} = 12[-5(1-x)^{-6}(-1)] = 60(1-x)^{-6} = \frac{60}{(1-x)^6} \quad f'''(0) = \frac{60}{(1-(0))^6} = \frac{60}{1^6} = 60$

$\vdots$

$f^{(k)}(x) = \frac{d^kf}{dx^k} = \left(\frac{(k+2)!}{2}\right)\left(\frac{1}{(1-x)^{k+3}}\right) \qquad f^{(k)}(0) = \left(\frac{(k+2)!}{2}\right)\left(\frac{1}{(1-(0))^{k+3}}\right) = \left(\frac{(k+2)!}{2}\right)$

30) continued

$$T(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 + \frac{f'''(0)}{3!}(x-(0))^3 + \cdots + \frac{f^{(k)}(0)}{k!}(x-(0))^k + \cdots$$

$$= (1) + (3)(x)^1 + \frac{(12)}{2}(x)^2 + \frac{(60)}{(2)(3)}(x)^3 + \cdots + \frac{\left(\frac{(k+2)!}{2}\right)}{k!}(x)^k + \cdots$$

$$= 1 + 3x + 6x^2 + 10x^3 + \cdots + \frac{(k+1)(k+2)}{2}x^k + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} x^k$$

---

32) $f(x) = 2^x$, $a = 1$

$f(1) = 2^{(1)} = 2$

$f'(x) = \frac{df}{dx} = 2^x(\ln 2)$ $\qquad f'(1) = 2^{(1)}(\ln 2) = 2(\ln 2)$

$f''(x) = \frac{d^2f}{dx^2} = 2^x(\ln 2)^2$ $\qquad f''(1) = 2^{(1)}(\ln 2)^2 = 2(\ln 2)^2$

$f'''(x) = \frac{d^3f}{dx^3} = 2^x(\ln 2)^3$ $\qquad f'''(1) = 2^{(1)}(\ln 2)^3 = 2(\ln 2)^3$

$\vdots$

$f^{(k)}(x) = \frac{d^kf}{dx^k} = 2^x(\ln 2)^k$ $\qquad f^{(k)}(1) = 2^{(1)}(\ln 2)^k = 2(\ln 2)^k$

$$T(x) = f(1) + f'(1)(x-(1))^1 + \frac{f''(1)}{2!}(x-(1))^2 + \frac{f'''(1)}{3!}(x-(1))^3 + \cdots + \frac{f^{(k)}(1)}{k!}(x-(1))^k + \cdots$$

$$= (2) + (2(\ln 2))(x-1)^1 + \frac{(2(\ln 2)^2)}{2}(x-1)^2 + \frac{(2(\ln 2)^3)}{(2)(3)}(x-1)^3 + \cdots + \frac{(2(\ln 2)^k)}{k!}(x-1)^k + \cdots$$

$$= 2 + 2(\ln 2)(x-1) + \frac{2(\ln 2)^2}{2!}(x-1)^2 + \frac{2(\ln 2)^3}{3!}(x-1)^3 + \cdots + \frac{2(\ln 2)^k}{k!}(x-1)^k + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{2(\ln 2)^k}{k!}(x-1)^k$$

34) $f(x) = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$, $a = 0$ $\quad f(0) = \sqrt{(0)+1} = \sqrt{1} = 1$

$$f'(x) = \frac{df}{dx} = \frac{1}{2}(x+1)^{\frac{-1}{2}}(1) = \frac{1}{2}(x+1)^{\frac{-1}{2}} = \frac{1}{2\sqrt{x+1}} \qquad f'(0) = \frac{1}{2\sqrt{(0)+1}} = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

$$f''(x) = \frac{d^2f}{dx^2} = \frac{1}{2}\left[\frac{-1}{2}(x+1)^{\frac{-3}{2}}(1)\right] = \frac{-1}{4}(x+1)^{\frac{-3}{2}} = \frac{-1}{4(\sqrt{x+1})^3} \qquad f''(0) = \frac{-1}{4(\sqrt{(0)+1})^3} = \frac{-1}{4(\sqrt{1})^3} = \frac{-1}{4}$$

$$f'''(x) = \frac{d^3f}{dx^3} = \frac{-1}{4}\left[\frac{-3}{2}(x+1)^{\frac{-5}{2}}(1)\right] = \frac{3}{8}(x+1)^{\frac{-5}{2}} = \frac{3}{8(\sqrt{x+1})^5} \qquad f'''(0) = \frac{3}{8(\sqrt{(0)+1})^5} = \frac{3}{8(\sqrt{1})^5} = \frac{3}{8}$$

$$f^{(4)}(x) = \frac{d^4f}{dx^4} = \frac{3}{8}\left[\frac{-5}{2}(x+1)^{\frac{-7}{2}}(1)\right] = \frac{-15}{16}(x+1)^{\frac{-7}{2}} = \frac{-15}{16(\sqrt{x+1})^7} \qquad f^{(4)}(0) = \frac{-15}{16(\sqrt{(0)+1})^7} = \frac{-15}{16(\sqrt{1})^7} = \frac{-15}{16}$$

$$T(x) = f(0) + f'(0)(x-(0))^1 + \frac{f''(0)}{2!}(x-(0))^2 + \frac{f'''(0)}{3!}(x-(0))^3 + \frac{f^{(4)}(0)}{4!}(x-(0))^4 + \dots$$

$$= (1) + \left(\frac{1}{2}\right)(x)^1 + \frac{\left(\frac{-1}{4}\right)}{2!}(x)^2 + \frac{\left(\frac{3}{8}\right)}{3!}(x)^3 + \frac{\left(\frac{-15}{16}\right)}{4!}(x)^4 + \dots$$

$$= 1 + \left(\frac{1}{2}\right)x - \left(\frac{1}{4}\right)\left(\frac{1}{2!}\right)x^2 + \left(\frac{3}{8}\right)\left(\frac{1}{3!}\right)x^3 - \left(\frac{15}{16}\right)\left(\frac{1}{4!}\right)x^4 + \dots$$

$$= 1 + \left\{\frac{(-1)^0}{(2)^1(1!)}x^1 + \frac{(-1)^1}{(2)^2(2!)}x^2 + \frac{(-1)^2(3)}{(2)^3(3!)}x^3 + \frac{(-1)^4(3)(5)}{(2)^4(4!)}x^4 + \dots\right\}$$

$$T(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots$$

36) $f(x) = (1 - x + x^2)e^x$

from table 10.1 in section 10.10, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

which converges absolutely on $(-\infty, \infty)$

The Maclaurin series generated by $f(x) = (1 - x + x^2)e^x$ is given by

$$M(x) = (1 - x + x^2)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) = 1 + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \cdots$$

which converges absolutely on $(-\infty, \infty)$

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38) $f(x) = x \sin^2 x$

from table 10.1 in section 10.10,

$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ which converges absolutely on $(-\infty, \infty)$

The Maclaurin series generated by $f(x) = x \sin^2 x$ is given by

$$M(x) = x\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right)^2 = x\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right)$$

$$= x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= x^3 - \frac{1}{3}x^5 + \frac{2}{45}x^7 + \cdots$$

which converges absolutely on $(-\infty, \infty)$

40) $f(x) = \frac{x^3}{1+2x}$

from table 10.1 in section 10.10, $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots$
which converges absolutely on $|x| < 1$ or $-1 < x < 1$

$\frac{1}{1+(2x)} = \sum_{n=0}^{\infty} (-1)^n (2x)^n = \sum_{n=0}^{\infty} (-1)^n (2^n)(x^n) = 1 - 2x + 4x^2 - 8x^3 + \dots$
which converges absolutely on $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$

The Maclaurin series generated by $f(x) = \frac{x^3}{1+2x} = (x^3)\left(\frac{1}{1+2x}\right)$
is given by

$$M(x) = (x^3)\left(\sum_{n=0}^{\infty} (-1)^n (2^n)(x^n)\right) = (x^3)\left(1 - 2x + 4x^2 - 8x^3 + \dots\right)$$
$$= x^3 - 2x^4 + 4x^5 - \dots$$

which converges absolutely on $|x| < \frac{1}{2}$
$-\frac{1}{2} < x < \frac{1}{2}$