Definition

A power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots . \qquad (1)$$

A power series about $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots \qquad (2)$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

Theorem 18 - The Convergence Theorem for Power Series

If the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.

Corollary to Theorem 18

The convergence of the series $\sum c_n (x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.

2. The series converges absolutely for every x ($R = \infty$).

3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the largest open interval where the series converges absolutely.
   $|x-a| < R$ or $a - R < x < a + R$.

2. If R is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b (see pages 626 to 628). Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. If R is finite, the series diverges for $|x-a| > R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.

Theorem 19 - Series Multiplication for Power Series

If $A(x)=\sum_{n=0}^{\infty} a_n x^n$ and $B(x)=\sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x|<R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k} \,,$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x|<R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n \,.$$

Theorem 20

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x|<R$ and f is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points x where $|f(x)|<R$.

Theorem 21 - Term-by-Term Differentiation

If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \qquad \text{on the interval} \qquad a-R<x<a+R\,.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

and so on. Each of these derived series converges at every point of the interval $a-R<x<a+R$.

Theorem 22 - Term-by-Term Integration

Suppose that

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a-R<x<a+R$ ($R>0$). Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a-R<x<a+R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a-R<x<a+R$.

2) $\sum_{n=0}^{\infty}(x+5)^n$ $|a_n| = u_n = (x+5)^n$ $u_{n+1} = (x+5)^{n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty}\left|\frac{(x+5)^{n+1}}{(x+5)^n}\right| = \lim_{n\to\infty}\left|\frac{((x+5)^n)((x+5)^1)}{(x+5)^n}\right| = \lim_{n\to\infty}|x+5| = |x+5|$$

$|x+5|<1$

$-1 < x+5 < 1$

$-6 < x < -4$

$|x|<1$

when $x=-6$, $\sum_{n=0}^{\infty}((-6)+5)^n = \sum_{n=0}^{\infty}(-1)^n$ which is an alternating divergent series

when $x=-4$, $\sum_{n=0}^{\infty}((-4)+5)^n = \sum_{n=0}^{\infty}(1)^n$ which is a divergent series

a) the radius is 1

the interval of convergence is $-6<x<-4$

b) the interval of absolute convergence is $-6<x<-4$

c) there are no values for which the series converges conditionally

4) $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$ $\quad |a_n| = u_n = \frac{(3x-2)^n}{n}$ $\quad u_{n+1} = \frac{(3x-2)^{n+1}}{n+1}$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

$$1 > \lim_{n \to \infty} \left| \frac{\frac{(3x-2)^{n+1}}{n+1}}{\frac{(3x-2)^n}{n}} \right| = \lim_{n \to \infty} \left| \left(\frac{(3x-2)^{n+1}}{n+1} \right) \left(\frac{n}{(3x-2)^n} \right) \right|$$

$$1 > \lim_{n \to \infty} \left| \left(\frac{((3x-2)^n)((3x-2)^1)}{n+1} \right) \left(\frac{n}{(3x-2)^n} \right) \right| = \lim_{n \to \infty} \left| \frac{(3x-2)\,n}{n+1} \right|$$

$$1 > \lim_{n \to \infty} |3x-2| \left| \frac{n}{n+1} \right| = |3x-2| \left| \lim_{n \to \infty} \frac{n}{n+1} \right| = |3x-2| \left| \lim_{n \to \infty} \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$1 > |3x-2| \left| \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}} \right| = |3x-2| \left| \frac{1}{1+0} \right| = |3x-2|$$

$|3x-2| < 1$

$-1 < 3x-2 < 1$

$1 < 3x < 3$

$\frac{1}{3} < x < 1$

$|3x| < 1$

$|x| < \frac{1}{3}$

when $x = \frac{1}{3}$, $\sum_{n=1}^{\infty} \frac{(3(\frac{1}{3})-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is the alternating harmonic series and is conditionally convergent.

when $x = 1$, $\sum_{n=1}^{\infty} \frac{(3(1)-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n}$ which is the divergent harmonic series.

a) the radius is $\frac{1}{3}$
the interval of convergence is $\frac{1}{3} \leq x < 1$

b) the interval of absolute convergence is $\frac{1}{3} < x < 1$

c) the series converges conditionally at $x = \frac{1}{3}$

8) $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$ $\qquad |a_n| = u_n = \frac{(x+2)^n}{n} \qquad u_{n+1} = \frac{(x+2)^{n+1}}{n+1}$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

$$1 > \lim_{n \to \infty} \left| \frac{\frac{(x+2)^{n+1}}{n+1}}{\frac{(x+2)^n}{n}} \right| = \lim_{n \to \infty} \left| \left(\frac{(x+2)^{n+1}}{n+1} \right) \left(\frac{n}{(x+2)^n} \right) \right| = \lim_{n \to \infty} \left| \left(\frac{((x+2)^n)((x+2)^1)}{n+1} \right) \left(\frac{n}{(x+2)^n} \right) \right|$$

$$1 > \lim_{n \to \infty} \left| \frac{(x+2)\, n}{n+1} \right| = \lim_{n \to \infty} |x+2| \left| \frac{n}{n+1} \right| = |x+2| \left| \lim_{n \to \infty} \frac{n}{n+1} \right| = |x+2| \left| \lim_{n \to \infty} \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} \right|$$

$$1 > |x+2| \left| \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n}} \right| = |x+2| \left| \frac{1}{1+0} \right| = |x+2|$$

$|x+2| < 1$

$-1 < x+2 < 1$

$-3 < x < -1$

$|x| < 1$

when $x = -3$, $\sum_{n=1}^{\infty} \frac{(-1)^n ((-3)+2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(1)^n}{n}$
which is the divergent harmonic series

when $x = -1$, $\sum_{n=1}^{\infty} \frac{(-1)^n ((-1)+2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n (1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
which is the alternating harmonic series and is conditionally convergent.

a) the radius is 1
the interval of convergence is $-3 < x \le -1$

b) the interval of absolute convergence is $-3 < x < -1$

c) the series converges conditionally at $x = -1$

10) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$ $|a_n| = u_n = \frac{(x-1)^n}{\sqrt{n}}$ $u_{n+1} = \frac{(x-1)^{n+1}}{\sqrt{n+1}}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty} \left|\frac{\frac{(x-1)^{n+1}}{\sqrt{n+1}}}{\frac{(x-1)^n}{\sqrt{n}}}\right| = \lim_{n\to\infty} \left|\left(\frac{(x-1)^{n+1}}{\sqrt{n+1}}\right)\left(\frac{\sqrt{n}}{(x-1)^n}\right)\right| = \lim_{n\to\infty} \left|\left(\frac{((x-1)^n)((x-1)^1)}{\sqrt{n+1}}\right)\left(\frac{\sqrt{n}}{(x-1)^n}\right)\right|$$

$$1 > \lim_{n\to\infty} \left|\frac{(x-1)\sqrt{n}}{\sqrt{n+1}}\right| = \lim_{n\to\infty} |x-1| \left|\frac{\sqrt{n}}{\sqrt{n+1}}\right| = |x-1| \lim_{n\to\infty} \left|\sqrt{\frac{n}{n+1}}\right| = |x-1| \left|\sqrt{\lim_{n\to\infty} \frac{n}{n+1}}\right|$$

$$1 > |x-1| \left|\sqrt{\lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}}\right| = |x-1| \left|\sqrt{\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}}\right| = |x-1| \left|\sqrt{\frac{1}{1+0}}\right| = |x-1|$$

$|x-1| < 1$

$-1 < x-1 < 1$

$0 < x < 2$

$|x| < 1$

when $x=0$, $\sum_{n=1}^{\infty} \frac{((0)-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ which is an alternating p-series and is conditionally convergent.

when $x=2$, $\sum_{n=1}^{\infty} \frac{((2)-1)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(1)^n}{\sqrt{n}}$ which is a divergent p-series because $p = \frac{1}{2} \leq 1$

a) the radius is 1

the interval of convergence is $0 \leq x < 2$

b) the interval of absolute convergence is $0 < x < 2$

c) the series converges conditionally at $x = 0$.

12) $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$ $\quad |a_n| = u_n = \frac{3^n x^n}{n!}$ $\quad u_{n+1} = \frac{3^{n+1} x^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty} \left|\frac{\frac{3^{n+1} x^{n+1}}{(n+1)!}}{\frac{3^n x^n}{n!}}\right| = \lim_{n\to\infty} \left|\left(\frac{3^{n+1} x^{n+1}}{(n+1)!}\right)\left(\frac{n!}{3^n x^n}\right)\right| = \lim_{n\to\infty} \left|\left(\frac{(3^n)(3^1)(x^n)(x^1)}{(n+1)\, n!}\right)\left(\frac{n!}{3^n x^n}\right)\right|$$

$$1 > \lim_{n\to\infty} \left|\frac{3x}{n+1}\right| = \lim_{n\to\infty} |3x|\left|\frac{1}{n+1}\right| = |3x|\left|\lim_{n\to\infty} \frac{1}{n+1}\right| = |3x||0| = 0$$

$0 < 1$ for all values of x

a) the radius is ∞

the series converges for all x $(-\infty, \infty)$

b) the series converges absolutely for all x $(-\infty, \infty)$

c) there are no values for which the series converge conditionally

14) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$ $\quad |a_n| = u_n = \frac{(x-1)^n}{n^3 3^n}$ $\quad u_{n+1} = \frac{(x-1)^{n+1}}{(n+1)^3 3^{n+1}}$

$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$

$$1 > \lim_{n \to \infty} \left| \frac{\frac{(x-1)^{n+1}}{(n+1)^3 3^{n+1}}}{\frac{(x-1)^n}{n^3 3^n}} \right| = \lim_{n \to \infty} \left| \left(\frac{(x-1)^{n+1}}{(n+1)^3 3^{n+1}} \right) \left(\frac{n^3 3^n}{(x-1)^n} \right) \right| = \lim_{n \to \infty} \left| \left(\frac{((x-1)^n)((x-1)^1)}{(n+1)^3 (3^n)(3^1)} \right) \left(\frac{n^3 3^n}{(x-1)^n} \right) \right|$$

$$1 > \lim_{n \to \infty} \left| \frac{(x-1) n^3}{3(n+1)^3} \right| = \lim_{n \to \infty} |x-1| \left| \frac{n^3}{3(n+1)^3} \right| = |x-1| \left| \lim_{n \to \infty} \frac{n^3}{3n^3 + 9n^2 + 9n + 3} \right|$$

$$1 > |x-1| \left| \lim_{n \to \infty} \frac{\frac{n^3}{n^3}}{\frac{3n^3}{n^3} + \frac{9n^2}{n^3} + \frac{9n}{n^3} + \frac{3}{n^3}} \right| = |x-1| \left| \lim_{n \to \infty} \frac{1}{3 + \frac{9}{n} + \frac{9}{n^2} + \frac{3}{n^3}} \right| = |x-1| \left| \frac{1}{3+0+0+0} \right| =$$

$$1 > |x-1| \left| \frac{1}{3} \right| = \frac{1}{3} |x-1|$$

$\frac{1}{3} |x-1| < 1$

$|x-1| < 3$

$-3 < x-1 < 3$

$-2 < x < 4$

$\frac{1}{3} |x| < 1$

$|x| < 3$

when $x = -2$, $\sum_{n=1}^{\infty} \frac{((-2)-1)^n}{n^3 3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n (3^n)}{n^3 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$
which is an alternating p-series and is absolutely convergent because $p = 3 > 1$

when $x = 4$, $\sum_{n=1}^{\infty} \frac{((4)-1)^n}{n^3 3^n} = \sum_{n=1}^{\infty} \frac{(3)^n}{n^3 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^3}$
which is an absolutely convergent p-series

a) the radius is 3
the interval of convergence is $-2 \le x \le 4$

b) the interval of absolute convergence is $-2 \le x \le 4$

c) there are no values for which the series converge conditionally

1b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n}+3}$ $\quad |a_n| = u_n = \frac{x^{n+1}}{\sqrt{n}+3} \quad u_{n+1} = \frac{x^{n+2}}{\sqrt{n+1}+3}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty} \left| \frac{\frac{x^{n+2}}{\sqrt{n+1}+3}}{\frac{x^{n+1}}{\sqrt{n}+3}} \right| = \lim_{n\to\infty} \left|\left(\frac{x^{n+2}}{\sqrt{n+1}+3}\right)\left(\frac{\sqrt{n}+3}{x^{n+1}}\right)\right| = \lim_{n\to\infty} \left|\left(\frac{(x)(x^{n+1})}{\sqrt{n+1}+3}\right)\left(\frac{\sqrt{n}+3}{x^{n+1}}\right)\right|$$

$$1 > \lim_{n\to\infty} \left|\frac{x(\sqrt{n}+3)}{\sqrt{n+1}+3}\right| = \lim_{n\to\infty} |x| \left|\frac{\sqrt{n}+3}{\sqrt{n+1}+3}\right| = |x| \left|\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}}+\frac{3}{\sqrt{n}}}{\frac{\sqrt{n+1}}{\sqrt{n}}+\frac{3}{\sqrt{n}}}\right|$$

$$1 > |x|\left|\lim_{n\to\infty} \frac{1+\frac{3}{\sqrt{n}}}{\sqrt{\frac{n}{n}+\frac{1}{n}}+\frac{3}{\sqrt{n}}}\right| = |x|\left|\lim_{n\to\infty} \frac{1+\frac{3}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}+\frac{3}{n}}\right| = |x|\left|\frac{1+0}{\sqrt{1+0}+0}\right| = |x|$$

$|x| < 1$

$-1 < x < 1$

$|x| < 1$

when $x=-1$, $\sum_{n=0}^{\infty} \frac{(-1)^n(-1)^{n+1}}{\sqrt{n}+3} = \sum_{n=0}^{\infty} \frac{(-1)^n(-1)^n(-1)^1}{\sqrt{n}+3} = \sum_{n=0}^{\infty} \frac{-1}{\sqrt{n}+3}$ which is a diverging series by the Direct Comparison Test using $\sum_{n=0}^{\infty} \frac{-1}{\sqrt{n}}$ which is a diverging p-series.

when $x=1$, $\sum_{n=0}^{\infty} \frac{(-1)^n(1)^n}{\sqrt{n}+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}+3}$ which is an alternating series and is conditionally convergent. Compare with $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

a) the radius is 1
the interval of convergence is $-1 < x \leq 1$

b) the interval of absolute convergence is $-1 < x < 1$

c) the series converges conditionally at $x=1$

18) $\sum_{n=0}^{\infty} \frac{n x^n}{4^n(n^2+1)}$ $\qquad |a_n| = u_n = \frac{n x^n}{4^n(n^2+1)} \qquad u_{n+1} = \frac{(n+1)x^{n+1}}{4^{n+1}((n+1)^2+1)}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty} \left| \frac{\frac{(n+1)x^{n+1}}{4^{n+1}((n+1)^2+1)}}{\frac{n x^n}{4^n(n^2+1)}} \right| = \lim_{n\to\infty} \left|\left(\frac{(n+1)x^{n+1}}{4^{n+1}((n+1)^2+1)}\right)\left(\frac{4^n(n^2+1)}{n x^n}\right)\right|$$

$$1 > \lim_{n\to\infty} \left|\left(\frac{(n+1)(x^n)(x^1)}{(4^n)(4^1)((n^2+2n+1)+1)}\right)\left(\frac{4^n(n^2+1)}{n x^n}\right)\right| = \lim_{n\to\infty} \left|\frac{x(n+1)(n^2+1)}{4n(n^2+2n+2)}\right|$$

$$1 > \lim_{n\to\infty} |x|\left|\frac{n^3+n^2+n+1}{4n^3+8n^2+2n}\right| = |x|\left|\lim_{n\to\infty} \frac{n^3+n^2+n+1}{4n^3+8n^2+2n}\right| = |x|\left|\lim_{n\to\infty} \frac{\frac{n^3}{n^3}+\frac{n^2}{n^3}+\frac{n}{n^3}+\frac{1}{n^3}}{\frac{4n^3}{n^3}+\frac{8n^2}{n^3}+\frac{2n}{n^3}}\right|$$

$$1 > |x|\left|\lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}}{4+\frac{8}{n}+\frac{2}{n^2}}\right| = |x|\left|\frac{1+0+0+0}{4+0+0}\right| = |x|\left|\frac{1}{4}\right| = |x|\frac{1}{4} = \frac{1}{4}|x|$$

$\frac{1}{4}|x| < 1$

$|x| < 4$

$-4 < x < 4$

$\frac{1}{4}|x| < 1$

$|x| < 4$

when $x=4$, $\sum_{n=0}^{\infty} \frac{n(4)^n}{4^n(n^2+1)} = \sum_{n=0}^{\infty} \frac{n}{n^2+1}$ which is a divergent series. Use $\sum_{n=0}^{\infty} \frac{1}{n}$ which is divergent and use the Limit Comparison Test.

when $x=-4$, $\sum_{n=0}^{\infty} \frac{n(-4)^n}{4^n(n^2+1)} = \sum_{n=0}^{\infty} \frac{n(-1)^n(4^n)}{4^n(n^2+1)} = \sum_{n=0}^{\infty} \frac{n(-1)^n}{n^2+1}$ which is an alternating series and is conditionally convergent.

a) the radius is 4
the interval of convergence is $-4 \le x < 4$

b) the interval of absolute convergence is $-4 < x < 4$

c) the series converges conditionally at $x = -4$

20) $\sum_{n=1}^{\infty} \sqrt[n]{n}\,(2x+5)^n$ $\quad |a_n| = u_n = \sqrt[n]{n}\,(2x+5)^n \quad u_{n+1} = \sqrt[n+1]{n+1}\,(2x+5)^{n+1}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty} \left|\frac{\sqrt[n+1]{n+1}\,(2x+5)^{n+1}}{\sqrt[n]{n}\,(2x+5)^n}\right| = \lim_{n\to\infty} \left|\frac{\sqrt[n+1]{n+1}\,\left((2x+5)^n\right)\left((2x+5)^1\right)}{\sqrt[n]{n}\,(2x+5)^n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt[n+1]{n+1}\,(2x+5)}{\sqrt[n]{n}}\right|$$

$$1 > \lim_{n\to\infty} |2x+5| \left|\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\right| = |2x+5| \left|\lim_{n\to\infty}\frac{\sqrt[n+1]{n+1}}{\sqrt[n]{n}}\right| = |2x+5|\left|\frac{\lim_{n\to\infty}\sqrt[n+1]{n+1}}{\lim_{n\to\infty}\sqrt[n]{n}}\right|$$

$\lim_{n\to\infty} \sqrt[n]{n} = ?$

$y = \sqrt[n]{n}$

$\ln y = \ln\left(\sqrt[n]{n}\right)$

$\ln y = \frac{\ln n}{n}$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{\ln n}}{\underset{+\infty}{n}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\ln y = 0$

$\Downarrow$

$y = e^0 = 1$

$\lim_{n\to\infty} \sqrt[n]{n} = 1$

$\lim_{n\to\infty} \sqrt[n+1]{n+1} = \lim_{t\to\infty} \sqrt[t]{t}$

let $n+1 = t$ $\quad = 1$

$$1 > |2x+5| \left|\frac{1}{1}\right| = |2x+5|$$

$|2x+5| < 1$

$-1 < 2x+5 < 1$

$-6 < 2x < -4$

$-3 < x < -2$

$|2x| < 1$

$2|x| < 1$

$x < \frac{1}{2}$

when $x = -3$, $\sum_{n=1}^{\infty} \sqrt[n]{n}\,(2(-3)+5)^n = \sum_{n=1}^{\infty} (-1)^n \sqrt[n]{n}$ which is an alternating series and it diverges because $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt[n]{n} = 1$

when $x = -2$, $\sum_{n=1}^{\infty} \sqrt[n]{n}\,(2(-2)+5)^n = \sum_{n=1}^{\infty} (1)^n \sqrt[n]{n} = \sum_{n=1}^{\infty} \sqrt[n]{n}$ which is a divergent series.

a) the radius is $\frac{1}{2}$

the interval of convergence is $-3 < x < -2$

b) the interval of absolute convergence is $-3 < x < -2$

c) there are no values for which the series converge conditionally

22) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n} (x-2)^n}{3n}$ $\quad |a_n| = u_n = \frac{3^{2n}(x-2)^n}{3n}$ $\quad u_{n+1} = \frac{3^{2(n+1)}(x-2)^{n+1}}{3(n+1)}$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

$$1 > \lim_{n\to\infty} \left| \frac{\frac{3^{2(n+1)}(x-2)^{n+1}}{3(n+1)}}{\frac{3^{2n}(x-2)^n}{3n}} \right| = \lim_{n\to\infty} \left| \left(\frac{3^{2n+2}(x-2)^{n+1}}{3(n+1)} \right) \left(\frac{3n}{3^{2n}(x-2)^n} \right) \right|$$

$$1 > \lim_{n\to\infty} \left| \left(\frac{(3^{2n})(3^2)((x-2)^n)((x-2)^1)}{3(n+1)} \right) \left(\frac{3n}{3^{2n}(x-2)^n} \right) \right| = \lim_{n\to\infty} \left| \frac{9n(x-2)}{n+1} \right|$$

$$1 > \lim_{n\to\infty} |x-2| \left| \frac{9n}{n+1} \right| = |x-2| \left| \lim_{n\to\infty} \frac{9n}{n+1} \right| = |x-2| \left| \lim_{n\to\infty} \frac{\frac{9n}{n}}{\frac{n}{n}+\frac{1}{n}} \right| = |x-2| \left| \lim_{n\to\infty} \frac{9}{1+\frac{1}{n}} \right|$$

$$1 > |x-2| \left| \frac{9}{1+0} \right| = |x-2| \, |9| = 9|x-2|$$

$9|x-2| < 1$

$|x-2| < \frac{1}{9}$

$\frac{-1}{9} < x-2 < \frac{1}{9}$

$\frac{17}{9} < x < \frac{19}{9}$

$9|x| < 1$

$|x| < \frac{1}{9}$

when $x = \frac{17}{9}$, $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n} ((\frac{17}{9})-2)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n} (\frac{-1}{9})^n}{3^n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n} (-1)^n (\frac{1}{3^2})^n}{3n} = \sum_{n=1}^{\infty} \frac{3^{2n}(\frac{1}{3^{2n}})}{3n} = \sum_{n=1}^{\infty} \frac{1}{3n}$

which is a divergent series. Use $\sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent p-series and use the Limit Comparison Test,

when $x = \frac{19}{9}$, $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n} ((\frac{19}{9})-2)^n}{3n} = \sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n} (\frac{1}{3^2})^n}{3n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$

which is an alternating series and is conditionally convergent.

a) the radius is $\frac{1}{9}$

the interval of convergence is $\frac{17}{9} < x \leq \frac{19}{9}$

b) the interval of absolute convergence is $\frac{17}{9} < x < \frac{19}{9}$

c) the series converges conditionally at $x = \frac{19}{9}$

24) $\sum_{n=1}^{\infty} (\ln n) x^{n}$ $\quad |a_n| = u_n = (\ln n) x^{n} \quad u_{n+1} = (\ln(n+1)) x^{n+1}$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1$$

$$1 > \lim_{n \to \infty} \left| \frac{(\ln(n+1)) x^{n+1}}{(\ln n) x^{n}} \right| = \lim_{n \to \infty} \left| \frac{(\ln(n+1))(x^{n})(x^{1})}{(\ln n) x^{n}} \right| = \lim_{n \to \infty} \left| \frac{x \ln(n+1)}{\ln n} \right|$$

$$1 > \lim_{n \to \infty} |x| \left| \frac{\ln(n+1)}{\ln n} \right| = |x| \left| \lim_{n \to \infty} \frac{\overset{+\infty}{\ln(n+1)}}{\underset{+\infty}{\ln n}} \right| \overset{L}{=} |x| \left| \lim_{n \to \infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} \right|$$

$$1 > |x| \left| \lim_{n \to \infty} \frac{\overset{+\infty}{n}}{\underset{+\infty}{n+1}} \right| \overset{L}{=} |x| \left| \lim_{n \to \infty} \frac{1}{1} \right| = |x| \left| \frac{1}{1} \right| = |x|$$

$|x| < 1$

$-1 < x < 1$

$|x| < 1$

when $x = 1$, $\sum_{n=1}^{\infty} (\ln n)(1)^{n} = \sum_{n=1}^{\infty} \ln n$ which is a divergent series because by the nth-Term Test for Divergence $\lim_{n \to \infty} (\ln n) = +\infty \neq 0$.

when $x = -1$, $\sum_{n=1}^{\infty} (\ln n)(-1)^{n}$ which is an alternating series and it diverges because $\lim_{n \to \infty} u_n = \lim_{n \to \infty} (\ln n) \neq 0$.

a) the radius is 1

the interval of convergence is $-1 < x < 1$

b) the interval of absolute convergence is $-1 < x < 1$

c) there are no values for which the series converge conditionally

26) $\sum_{n=0}^{\infty} n!\,(x-4)^n$ $\quad |a_n| = u_n = n!\,(x-4)^n \quad u_{n+1} = (n+1)!\,(x-4)^{n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty}\left|\frac{(n+1)!\,(x-4)^{n+1}}{n!\,(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)n!\,((x-4)^n)((x-4)^1)}{n!\,(x-4)^n}\right|$$

$$1 > \lim_{n\to\infty}\left|(n+1)(x-4)\right| = \lim_{n\to\infty}|x-4||n+1| = |x-4|\left|\lim_{n\to\infty}(n+1)\right|$$

$1 > |x-4|\left|\lim_{n\to\infty}(n+1)\right|$ $\quad \lim_{n\to\infty}(n+1) = +\infty$ so the only value that will satisfy the inequality is when $x=4$

when $x=4$, $\sum_{n=0}^{\infty} n!\,((4)-4)^n = \sum_{n=0}^{\infty} n!\,(0)^n = \sum_{n=0}^{\infty} 0 = 0$

a) the radius is 0

the series converges only for $x=4$

b) the series absolutely converges only for $x=4$

c) there are no values for which the series converges conditionally

28) $\sum_{n=0}^{\infty}(-2)^n(n+1)(x-1)^n$ $\quad |a_n| = u_n = 2^n(n+1)(x-1)^n$

$u_{n+1} = 2^{n+1}((n+1)+1)(x-1)^{n+1}$

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$

$$1 > \lim_{n\to\infty}\left|\frac{2^{n+1}((n+1)+1)(x-1)^{n+1}}{2^n(n+1)(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(2^n)(2^1)(n+2)((x-1)^n)((x-1)^1)}{2^n(n+1)(x-1)^n}\right|$$

$$1 > \lim_{n\to\infty}\left|\frac{2(n+2)(x-1)}{n+1}\right| = \lim_{n\to\infty}|x-1|\left|\frac{2n+2}{n+1}\right| = |x-1|\left|\lim_{n\to\infty}\frac{2n+2}{n+1}\right|$$

$$1 > |x-1|\left|\lim_{n\to\infty}\frac{\frac{2n}{n}+\frac{2}{n}}{\frac{n}{n}+\frac{1}{n}}\right| = |x-1|\left|\lim_{n\to\infty}\frac{2+\frac{2}{n}}{1+\frac{1}{n}}\right| = |x-1|\left|\frac{2+0}{1+0}\right| = |x-1||2| = 2|x-1|$$

$2|x-1|<1$

$|x-1|<\frac{1}{2}$

$\frac{-1}{2}<x-1<\frac{1}{2}$

$\frac{1}{2}<x<\frac{3}{2}$

$2|x|<1$

$|x|<\frac{1}{2}$

when $x=\frac{1}{2}$, $\sum_{n=0}^{\infty}(-2)^n(n+1)((\frac{1}{2})-1)^n = \sum_{n=0}^{\infty}(-2)^n(n+1)(\frac{-1}{2})^n$
$= \sum_{n=0}^{\infty}(n+1)$

which is a divergent series.

when $x=\frac{3}{2}$, $\sum_{n=0}^{\infty}(-2)^n(n+1)((\frac{3}{2})-1)^n = \sum_{n=0}^{\infty}(-2)^n(n+1)(\frac{1}{2})^n$
$= \sum_{n=0}^{\infty}(-1)^n(n+1)$

which is an alternating series and is divergent because $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(n+1) \neq 0$

a) the radius is $\frac{1}{2}$

the interval of convergence is $\frac{1}{2}<x<\frac{3}{2}$

b) the interval of absolute convergence is $\frac{1}{2}<x<\frac{3}{2}$

c) there are no values for which the series converges conditionally

30) $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$ $\qquad |a_n| = u_n = \frac{x^n}{n \ln n} \qquad u_{n+1} = \frac{x^{n+1}}{(n+1)\ln(n+1)}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)\ln(n+1)}}{\frac{x^n}{n \ln n}} \right| = \lim_{n\to\infty} \left| \left(\frac{x^{n+1}}{(n+1)\ln(n+1)}\right)\left(\frac{n \ln n}{x^n}\right) \right|$$

$$1 > \lim_{n\to\infty} \left| \left(\frac{(x^n)(x^1)}{(n+1)\ln(n+1)}\right)\left(\frac{n \ln n}{x^n}\right) \right| = \lim_{n\to\infty} \left| \frac{x \; n \ln n}{(n+1)\ln(n+1)} \right| = \lim_{n\to\infty} |x| \left| \left(\frac{n}{n+1}\right)\left(\frac{\ln n}{\ln(n+1)}\right) \right|$$

$$1 > |x| \left| \left(\lim_{n\to\infty} \frac{n}{n+1}\right)\left(\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)}\right) \right| = |x| \left|(1)(1)\right| = |x|$$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{\ln n}}{\underset{+\infty}{\ln(n+1)}} \overset{L}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{\overset{+\infty}{n+1}}{\underset{+\infty}{n}} \overset{L}{=} \lim_{n\to\infty} \frac{1}{1} = \frac{1}{1} = 1$$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{n}}{\underset{+\infty}{n+1}} \overset{L}{=} \lim_{n\to\infty} \frac{1}{1} = \frac{1}{1} = 1$$

$|x| < 1$ $\qquad$ when $x = 1$, $\sum_{n=2}^{\infty} \frac{(1)^n}{n \ln n} = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $\qquad f(x) = \frac{1}{x \ln x}$

$-1 < x < 1$ $\qquad \int \frac{1}{x \ln x} dx = \int \frac{1}{p} dp = \ln|p| + C = \ln|\ln x| + C$

$|x| < 1$ $\qquad p = \ln x$

$dp = \frac{1}{x} dx$

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{v\to\infty} \int_2^{v} \frac{1}{x \ln x} dx = \lim_{v\to\infty} \left[\ln|\ln x| + C\right]_2^{v}$$

$$= \lim_{v\to\infty} \left\{ \left[\ln|\ln v| + C\right] - \left[\ln|\ln(2)| + C\right] \right\} = +\infty$$

By the Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

when $x = -1$, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ which is an alternating series and is conditionally convergent

a) the radius is 1

the interval of convergence is $-1 \le x < 1$

b) the interval of absolute convergence is $-1 < x < 1$

c) the series converges conditionally at $x = -1$

32) $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$ $\quad |a_n| = u_n = \frac{(3x+1)^{n+1}}{2n+2}$ $\quad u_{n+1} = \frac{(3x+1)^{(n+1)+1}}{2(n+1)+2}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty} \left| \frac{\frac{(3x+1)^{(n+1)+1}}{2(n+1)+2}}{\frac{(3x+1)^{n+1}}{2n+2}} \right| = \lim_{n\to\infty} \left| \left(\frac{(3x+1)^{n+2}}{2n+4}\right)\left(\frac{2n+2}{(3x+1)^{n+1}}\right)\right|$$

$$1 > \lim_{n\to\infty} \left|\left(\frac{((3x+1)^{n+1})((3x+1)^{1})}{2n+4}\right)\left(\frac{2n+2}{(3x+1)^{n+1}}\right)\right| = \lim_{n\to\infty} \left|\frac{(3x+1)\,(2n+2)}{2n+4}\right|$$

$$1 > |3x+1| \left|\lim_{n\to\infty} \frac{\overset{+\infty}{2n+2}}{\underset{+\infty}{2n+4}}\right| \overset{L}{=} |3x+1|\left|\lim_{n\to\infty} \frac{2}{2}\right| = |3x+1|\left|\frac{2}{2}\right| = |3x+1|\,|1| = |3x+1|$$

$|3x+1| < 1$

$-1 < 3x+1 < 1$

$-2 < 3x < 0$

$\frac{-2}{3} < x < 0$

$|3x| < 1$

$3|x| < 1$

$|x| < \frac{1}{3}$

when $x=0$, $\sum_{n=1}^{\infty} \frac{(3(0)+1)^{n+1}}{2n+2} = \sum_{n=1}^{\infty} \frac{1}{2n+2}$ which is a divergent series

when $x = \frac{-2}{3}$, $\sum_{n=1}^{\infty} \frac{(3(\frac{-2}{3})+1)^{n+1}}{2n+2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$ which is an alternating series and is conditionally convergent.

a) the radius is $\frac{1}{3}$

the interval of convergence is $\frac{-2}{3} \leq x < 0$

b) the interval of absolute convergence is $\frac{-2}{3} < x < 0$

c) the series converges conditionally at $x = \frac{-2}{3}$

34) $\sum_{n=1}^{\infty} \frac{(3)(5)(7)\cdots(2n+1)}{n^2 2^n} x^{n+1}$

$|a_n| = u_n = \frac{(3)(5)(7)\cdots(2n+1)}{n^2 2^n} x^{n+1}$

$u_{n+1} = \frac{(3)(5)(7)\cdots(2(n+1)+1)}{(n+1)^2 2^{n+1}} x^{(n+1)+1}$

$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$

$$1 > \lim_{n\to\infty} \left| \frac{\frac{(3)(5)(7)\cdots(2(n+1)+1)\, x^{(n+1)+1}}{(n+1)^2 2^{n+1}}}{\frac{(3)(5)(7)\cdots(2n+1)\, x^{n+1}}{n^2 2^n}} \right| = \lim_{n\to\infty} \left| \left(\frac{(3)(5)(7)\cdots(2(n+1)+1)\, x^{(n+1)+1}}{(n+1)^2 2^{n+1}}\right)\left(\frac{n^2 2^n}{(3)(5)(7)\cdots(2n+1)}\right) \right|$$

$$1 > \lim_{n\to\infty} \left| \left(\frac{(3)(5)(7)\cdots(2n+1)(2(n+1)+1)\,(x^n)(x^1)}{(n+1)^2 (2^n)(2^1)}\right)\left(\frac{n^2 2^n}{(3)(5)(7)\cdots(2n+1)}\right) \right| = \lim_{n\to\infty} \left|\frac{x\,(2n+3)n^2}{2\,(n+1)^2}\right|$$

$$1 > \lim_{n\to\infty} |x| \left|\frac{2n^3+3n^2}{2n^2+4n+2}\right| = |x| \left|\lim_{n\to\infty} \frac{2n^3+3n^2}{2n^2+4n+2}\right|$$

$$\lim_{n\to\infty} \frac{\overset{+\infty}{2n^3+3n^2}}{\underset{+\infty}{2n^2+4n+2}} \overset{L}{=} \lim_{n\to\infty} \frac{\overset{+\infty}{6n^2+6n}}{\underset{+\infty}{4n+4}} \overset{L}{=} \lim_{n\to\infty} \frac{12n+6}{4} = +\infty$$ so the only value that will satisfy the inequality is when $x = 0$

when $x=0$, $\sum_{n=1}^{\infty} \frac{(3)(5)(7)\cdots(2n+1)}{n^2 2^n}(0)^{n+1} = \sum_{n=1}^{\infty} 0 = 0$

a) the radius is 0

the series converges only for $x = 0$

b) the series absolutely converges only for $x = 0$

c) there are no values for which the series converges conditionally.

36) $\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})(x-3)^n = \sum_{n=1}^{\infty}\left(\frac{(\sqrt{n+1}-\sqrt{n})(x-3)^n}{1}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)$

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$

$$= \sum_{n=1}^{\infty}\frac{\{(n+1)-n\}(x-3)^n}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(x-3)^n}{\sqrt{n+1}+\sqrt{n}}$$

$$|a_n| = u_n = \frac{(x-3)^n}{\sqrt{n+1}+\sqrt{n}} \quad u_{n+1} = \frac{(x-3)^{n+1}}{\sqrt{(n+1)+1}+\sqrt{n+1}}$$

$$1 > \lim_{n\to\infty}\left|\frac{\frac{(x-3)^{n+1}}{\sqrt{(n+1)+1}+\sqrt{n+1}}}{\frac{(x-3)^n}{\sqrt{n+1}+\sqrt{n}}}\right| = \lim_{n\to\infty}\left|\left(\frac{(x-3)^{n+1}}{\sqrt{n+2}+\sqrt{n+1}}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{(x-3)^n}\right)\right|$$

$$1 > \lim_{n\to\infty}\left|\left(\frac{((x-3)^n)((x-3)^1)}{\sqrt{n+2}+\sqrt{n+1}}\right)\left(\frac{\sqrt{n+1}+\sqrt{n}}{(x-3)^n}\right)\right| = \lim_{n\to\infty}\left|\frac{(x-3)(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+2}+\sqrt{n+1}}\right|$$

$$1 > \lim_{n\to\infty}|x-3|\left|\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}}\right| = |x-3|\left|\lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}}\right| = |x-3||1| = |x-3|$$

$$\lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n+1}} = \lim_{n\to\infty}\frac{\frac{\sqrt{n+1}}{\sqrt{n}}+\frac{\sqrt{n}}{\sqrt{n}}}{\frac{\sqrt{n+2}}{\sqrt{n}}+\frac{\sqrt{n+1}}{\sqrt{n}}} = \lim_{n\to\infty}\frac{\sqrt{\frac{n}{n}+\frac{1}{n}}+\sqrt{\frac{n}{n}}}{\sqrt{\frac{n}{n}+\frac{2}{n}}+\sqrt{\frac{n}{n}+\frac{1}{n}}} = \lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}}+\sqrt{1}}{\sqrt{1+\frac{2}{n}}+\sqrt{1+\frac{1}{n}}}$$

$$= \frac{\sqrt{1+0}+\sqrt{1}}{\sqrt{1+0}+\sqrt{1+0}} = \frac{1+1}{1+1} = \frac{2}{2} = 1$$

$|x-3|<1$

$-1<x-3<1$

$2<x<4$

$|x|<1$

when $x=4$, $\sum_{n=1}^{\infty}\frac{((4)-3)^n}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(1)^n}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$

which is a divergent series. Use $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}}$ which diverges and compare with either Direct or Limit Test

when $x=2$, $\sum_{n=1}^{\infty}\frac{((2)-3)^n}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}+\sqrt{n}}$ which is an alternating series and is conditionally convergent

a) the radius is 1

the interval of convergence is $2 \le x < 4$

b) the interval of absolute convergence is $2 < x < 4$

c) the series converges conditionally at $x=2$

38) $\sum_{n=1}^{\infty}\left(\frac{(2)(4)(6)\cdots(2n)}{(2)(5)(8)\cdots(3n-1)}\right)^2 x^n$ $\quad |a_n| = u_n = \left(\frac{(2)(4)(6)\cdots(2n)}{(2)(5)(8)\cdots(3n-1)}\right)^2 x^n$

$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1$ $\quad u_{n+1} = \left(\frac{(2)(4)(6)\cdots(2(n+1))}{(2)(5)(8)\cdots(3(n+1)-1)}\right)^2 x^{n+1}$

$$1 > \lim_{n\to\infty}\left|\frac{\left(\frac{(2)(4)(6)\cdots(2(n+1))}{(2)(5)(8)\cdots(3(n+1)-1)}\right)^2 x^{n+1}}{\left(\frac{(2)(4)(6)\cdots(2n)}{(2)(5)(8)\cdots(3n-1)}\right)^2 x^n}\right| = \lim_{n\to\infty}\left|\left(\left(\frac{(2)(4)(6)\cdots(2n+2)}{(2)(4)(8)\cdots(3n+2)}\right)^2 x^{n+1}\right)\left(\left(\frac{(2)(5)(8)\cdots(3n-1)}{(2)(4)(6)\cdots(2n)}\right)^2\frac{1}{x^n}\right)\right|$$

$$1 > \lim_{n\to\infty}\left|\left(\left(\frac{(2)(4)(6)\cdots(2n)(2n+2)}{(2)(5)(8)\cdots(3n-1)(3n+2)}\right)^2\left(\frac{(x^n)(x^1)}{1}\right)\right)\left(\left(\frac{(2)(5)(8)\cdots(3n-1)}{(2)(4)(6)\cdots(2n)}\right)^2\left(\frac{1}{x^n}\right)\right)\right|$$

$$1 > \lim_{n\to\infty}\left|\frac{x}{1}\left(\frac{2n+2}{3n+2}\right)^2\right| = \lim_{n\to\infty}|x|\left|\left(\frac{2n+2}{3n+2}\right)^2\right| = |x|\left|\lim_{n\to\infty}\left(\frac{2n+2}{3n+2}\right)^2\right|$$

$$1 > |x|\left|\left(\lim_{n\to\infty}\frac{2n+2}{3n+2}\right)^2\right| = |x|\left|\left(\lim_{n\to\infty}\frac{\frac{2n}{n}+\frac{2}{n}}{\frac{3n}{n}+\frac{2}{n}}\right)^2\right| = |x|\left|\left(\lim_{n\to\infty}\frac{2+\frac{2}{n}}{3+\frac{2}{n}}\right)^2\right|$$

$$1 > |x|\left|\left(\frac{2+0}{3+0}\right)^2\right| = |x|\left|\left(\frac{2}{3}\right)^2\right| = |x|\left|\frac{4}{9}\right| = \frac{4}{9}|x|$$

$\frac{4}{9}|x| < 1 \Rightarrow |x| < \frac{9}{4}$ the radius is $R = \frac{9}{4}$

---

40) $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2} x^n$ {Hint: Apply the Root Test}

$\lim_{n\to\infty}\sqrt[n]{u_n} < 1$

$|a_n| = u_n = \left(\frac{n}{n+1}\right)^{n^2} x^n$

$$\begin{array}{r} 1 \\ n+1\,\overline{)\,n+0} \\ -(n+1) \\ \hline -1 \end{array}$$

$$1 > \lim_{n\to\infty}\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2} x^n} = \lim_{n\to\infty}\left(\sqrt[n]{x^n}\ \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}\right) = |x|\left|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\right|$$

$\frac{n}{n+1} = 1 + \frac{(-1)}{n+1}$

$$1 > |x|\left|\lim_{n\to\infty}\left(1+\frac{(-1)}{n+1}\right)^n\right| = |x|\left|e^{-1}\right| = \frac{1}{e}|x|$$

$\frac{1}{e}|x| < 1 \Rightarrow |x| < e$ the radius is $R = e$

42) $\sum_{n=0}^{\infty}(e^x-4)^n$ $\quad |a_n| = u_n = (e^x-4)^n \quad u_{n+1} = (e^x-4)^{n+1}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$$

$$1>\lim_{n\to\infty}\left|\frac{(e^x-4)^{n+1}}{(e^x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{((e^x-4)^n)((e^x-4)^1)}{(e^x-4)^n}\right| = \lim_{n\to\infty}\left|e^x-4\right| = \left|e^x-4\right|$$

$|e^x-4|<1$

$-1<e^x-4<1$

$3<e^x<5$

$\ln 3 < x < \ln 5$

when $x=\ln 3$, $\sum_{n=0}^{\infty}(e^{(\ln 3)}-4)^n = \sum_{n=0}^{\infty}(3-4)^n = \sum_{n=0}^{\infty}(-1)^n$

which is divergent series

when $x=\ln 5$, $\sum_{n=0}^{\infty}(e^{(\ln 5)}-4)^n = \sum_{n=0}^{\infty}(5-4)^n = \sum_{n=0}^{\infty}(1)^n$

which is divergent series

$$\sum_{n=0}^{\infty}(e^x-4)^n = (e^x-4)^0+(e^x-4)^1+(e^x-4)^2+\cdots$$

when $\ln 3 < x < \ln 5$, this is a convergent geometric series with $a=1$ and $r=(e^x-4)$ and the sum is

$$\frac{a}{1-r} = \frac{(1)}{1-(e^x-4)} = \frac{1}{5-e^x}.$$

---

44) $\sum_{n=0}^{\infty}\frac{(x+1)^{2n}}{9^n}$ $\quad |a_n| = u_n = \frac{(x+1)^{2n}}{9^n} \quad u_{n+1} = \frac{(x+1)^{2(n+1)}}{9^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$$

$$1>\lim_{n\to\infty}\left|\frac{\frac{(x+1)^{2(n+1)}}{9^{n+1}}}{\frac{(x+1)^{2n}}{9^n}}\right| = \lim_{n\to\infty}\left|\left(\frac{(x+1)^{2n+2}}{9^{n+1}}\right)\left(\frac{9^n}{(x+1)^{2n}}\right)\right| = \lim_{n\to\infty}\left|\left(\frac{((x+1)^{2n})((x+1)^2)}{(9^n)(9^1)}\right)\left(\frac{9^n}{(x+1)^{2n}}\right)\right|$$

44) continued

$$1 > \lim_{n\to\infty}\left|\frac{(x+1)^2}{9}\right| = \left|\frac{(x+1)^2}{9}\right| = \frac{|(x+1)^2|}{9}$$

$$\frac{|(x+1)^2|}{9} < 1$$

$$|(x+1)^2| < 9$$

$$|x+1| < 3$$

$$-3 < x+1 < 3$$

$$-4 < x < 2$$

when $x=2$, $\sum_{n=0}^{\infty}\frac{((2)+1)^{2n}}{9^n} = \sum_{n=0}^{\infty}\frac{(3)^{2n}}{9^n} = \sum_{n=0}^{\infty}\frac{(3^2)^n}{9^n} = \sum_{n=0}^{\infty}\frac{9^n}{9^n}$

$= \sum_{n=0}^{\infty} 1$ which is a divergent series

when $x=-4$, $\sum_{n=0}^{\infty}\frac{((-4)+1)^{2n}}{9^n} = \sum_{n=0}^{\infty}\frac{(-3)^{2n}}{9^n} = \sum_{n=0}^{\infty}\frac{((-3)^2)^n}{9^n} = \sum_{n=0}^{\infty}\frac{9^n}{9^n}$

$= \sum_{n=0}^{\infty} 1$ which is a divergent series

$$\sum_{n=0}^{\infty}\frac{(x+1)^{2n}}{9^n} = \sum_{n=0}^{\infty}\frac{((x+1)^2)^n}{(3^2)^n} = \sum_{n=0}^{\infty}\left(\frac{(x+1)^2}{3^2}\right)^n = \sum_{n=0}^{\infty}\left(\left(\frac{x+1}{3}\right)^2\right)^n$$

$$= \left(\left(\frac{x+1}{3}\right)^2\right)^0 + \left(\left(\frac{x+1}{3}\right)^2\right)^1 + \left(\left(\frac{x+1}{3}\right)^2\right)^2 + \cdots$$

when $-4 < x < 2$, this is a convergent geometric series with $a=1$ and $r = \left(\frac{x+1}{3}\right)^2$ and the sum is

$$\frac{a}{1-r} = \frac{(1)}{1-\left(\frac{x+1}{3}\right)^2} = \frac{1}{1-\frac{(x+1)^2}{9}} = \left(\frac{\frac{1}{1}}{\frac{1}{1}-\frac{(x+1)^2}{9}}\right)\left(\frac{\frac{9}{1}}{\frac{9}{1}}\right) = \frac{9}{9-(x+1)^2}$$

$$= \frac{9}{9-(x^2+2x+1)} = \frac{9}{8-2x-x^2}$$

46) $\sum_{n=0}^{\infty} (\ln x)^n$ $\quad |a_n| = u_n = (\ln x)^n \quad u_{n+1} = (\ln x)^{n+1}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty} \left|\frac{(\ln x)^{n+1}}{(\ln x)^n}\right| = \lim_{n\to\infty} \left|\frac{((\ln x)^n)((\ln x)^1)}{(\ln x)^n}\right| = \lim_{n\to\infty} |\ln x| = |\ln x|$$

$|\ln x| < 1$ $\quad$ when $x = e$, $\sum_{n=0}^{\infty} (\ln(e))^n = \sum_{n=0}^{\infty} (1)^n$ which is a

$-1 < \ln x < 1$ $\quad$ divergent series

$e^{-1} = \frac{1}{e} < x < e$ $\quad$ when $x = e^{-1}$, $\sum_{n=0}^{\infty} (\ln(e^{-1}))^n = \sum_{n=0}^{\infty} (-1)^n$ which is a

divergent series

$$\sum_{n=0}^{\infty} (\ln x)^n = (\ln x)^0 + (\ln x)^1 + (\ln x)^2 + \cdots$$

when $\frac{1}{e} < x < e$, this is a convergent geometric series with $a = 1$ and $r = \ln x$ and the sum is

$$\frac{a}{1-r} = \frac{(1)}{1-(\ln x)} = \frac{1}{1-\ln x}$$

---

48) $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2}\right)^n$ $\quad |a_n| = u_n = \left(\frac{x^2-1}{2}\right)^n \quad u_{n+1} = \left(\frac{x^2-1}{2}\right)^{n+1}$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1$$

$$1 > \lim_{n\to\infty} \left|\frac{\left(\frac{x^2-1}{2}\right)^{n+1}}{\left(\frac{x^2-1}{2}\right)^n}\right| = \lim_{n\to\infty} \left|\left(\frac{x^2-1}{2}\right)^{n+1}\left(\frac{2}{x^2-1}\right)^n\right| = \lim_{n\to\infty} \left|\left(\left(\frac{x^2-1}{2}\right)^n\right)\left(\left(\frac{x^2-1}{2}\right)^1\right)\left(\frac{2}{x^2-1}\right)^n\right|$$

$$1 > \lim_{n\to\infty} \left|\frac{x^2-1}{2}\right| = \left|\frac{x^2-1}{2}\right| = \frac{|x^2-1|}{2}$$

48) continued

$$\frac{|x^2-1|}{2} < 1$$

$$|x^2-1| < 2$$

$$(x^2-1) < 2$$

$$x^2-1 < 2$$

$$x^2-1<2$$

$$x^2-3<0$$

$$(x+\sqrt{3})(x-\sqrt{3})<0$$

$x+\sqrt{3}=0$ | $x-\sqrt{3}=0$

$x=-\sqrt{3}$ | $x=\sqrt{3}$

$-\sqrt{3}$ $\sqrt{3}$

	$(-\infty,-\sqrt{3})$	$(-\sqrt{3},\sqrt{3})$	$(\sqrt{3},\infty)$
$(x+\sqrt{3})$	neg	POS	POS
$(x-\sqrt{3})$	neg	neg	POS
$(x+\sqrt{3})(x-\sqrt{3})$	POS	neg	POS

$$-\sqrt{3}<x<\sqrt{3}$$

when $x=-\sqrt{3}$, $\sum_{n=0}^{\infty}\left(\frac{(-\sqrt{3})^2-1}{2}\right)^n=\sum_{n=0}^{\infty}\left(\frac{3-1}{2}\right)^n=\sum_{n=0}^{\infty}\left(\frac{2}{2}\right)^n=\sum_{n=0}^{\infty}(1)^n$

which is a divergent series.

When $x=\sqrt{3}$, $\sum_{n=0}^{\infty}\left(\frac{(\sqrt{3})^2-1}{2}\right)^n=\sum_{n=0}^{\infty}\left(\frac{3-1}{2}\right)^n=\sum_{n=0}^{\infty}\left(\frac{2}{2}\right)^n=\sum_{n=0}^{\infty}(1)^n$

which is a divergent series

$$\sum_{n=0}^{\infty}\left(\frac{x^2-1}{2}\right)^n=\left(\frac{x^2-1}{2}\right)^0+\left(\frac{x^2-1}{2}\right)^1+\left(\frac{x^2-1}{2}\right)^2+\cdots$$

when $-\sqrt{3}<x<\sqrt{3}$, this is a convergent geometric series with $a=1$ and $r=\frac{x^2-1}{2}$ and the sum is

$$\frac{a}{1-r}=\frac{(1)}{1-\left(\frac{x^2-1}{2}\right)}=\left(\frac{\frac{1}{1}}{\frac{1}{1}-\frac{x^2-1}{2}}\right)\left(\frac{\frac{2}{1}}{\frac{2}{1}}\right)=\frac{2}{2-(x^2-1)}=\frac{2}{3-x^2}$$